

Risk Premia in Structured Credit Derivatives

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Abstract

During the past couple of years much research effort has been devoted to explaining the spread of corporate bonds over Treasuries. On the other hand, relatively little is known about the spread components of structured credit products. This paper shows that such securities compensate investors for expected losses due to defaults, *pure* jump-to-default risk, correlation risk, as well as the risk of firm-specific and market-wide adverse changes in credit conditions. We provide a framework that allows a decomposition of "structured" credit spreads, and we apply this decomposition to CDX index tranches.

Keywords: credit risk, correlated default, structured credit derivatives, affine jump diffusion, tranche spread decomposition, portfolio loss decomposition

*Department of Statistics, Stanford University. All comments are welcome via email: andreas@eckner.com. The source code of the implementation is available at <http://www.eckner.com/research.html>. I would like to thank Xiaowei Ding, Kay Giesecke, Tze Leung Lai, Allan Mortensen and George Papanicolaou for helpful comments and remarks, and especially Darrell Duffie for frequent discussions. I am grateful to Citi, Markit and Morgan Stanley for providing historical CDX index and tranche spreads, as well as Barclays Capital and Markit for providing historical CDS spread data.

1 Introduction

We focus on the investment-grade CDX family of structured credit products, for their entire trading history up to November 2006. These products allocate default losses on a portfolio of US corporate debt of 125 firms, equally weighted, to a list of tranches. The first-loss, or “equity” tranche, for example, is allocated default losses as they occur, up to the 3% of the total notional amount of the 125-name portfolio. The other tranches cover losses, respectively between 3 and 7% of notional (“junior mezzanine”), between 7 and 10% of notional (“mezzanine”), between 10 and 15% of notional (“senior mezzanine”), between 15 and 30% of notional (“senior”), and between 30 and 100% of the notional (“super senior”).

We find that investors in the CDX first-loss tranche on average bear about 93.3% of the expected losses on the underlying portfolio of corporate debt, and receive about 77.0% of the total compensation for bearing portfolio losses. On the other hand, investors in the senior tranche, on average, bear about 0.2% of expected default losses on the underlying portfolio, and receive about 2% of the total compensation. In terms of the ratio, on average, of compensation rate per unit of expected loss, the first-loss piece carries a risk premium of 3.9, whereas the most senior tranches offer a risk premium of about 50. This extreme difference in risk premia is natural given that the most senior tranches bear losses only when there is a significant “meltdown” in corporate performance. As opposed to the finding of Coval, Jurek, and Stafford (2007) that investors do not demand much compensation for this extremely systematic risk, there does seem to be evidence of significant compensation for the systematic nature of this risk.

Using the methods developed in this paper, we are able to break down the compensation for bearing default risk into several components: compensation for actual expected losses, systematic risk, firm-specific risk, correlation risk, and *pure* jump-to-default (JTD) risk.¹ The various tranches have different rates of compensation per unit of expected loss because they carry these various types of risks in different proportions to each other. For example, the equity tranche is allocated 81.7% of the total compensation for firm-specific risk in the underlying portfolio of debt, but carries only 32.0% of the total compensation for systematic risk. The senior tranche, on the other hand, is allocated only 0.2% of the total compensation for firm-specific risk, but carries a relatively large fraction, 9.9%, of the total compensation for systematic risk.

For the 5-year CDX.NA.IG index on November 1, 2006, Figure 1 shows for each tranche the fraction of its compensation that can be attributed to each of the five risk

¹The latter component is sometimes called timing risk, or portfolio sampling risk. Compensation for correlation risk premium and *pure* jump-to-default risk would not be separately identifiable in a univariate setting. Hence, in recent research on the components of corporate bond and CDS spreads, the sum of these two risk premia is simply called *the* jump-to-default risk premium. See for example Driessen (2005), Berndt, Douglas, Duffie, Ferguson, and Schranz (2005), Amato (2005), Amato and Remolona (2005), and Saita (2006).

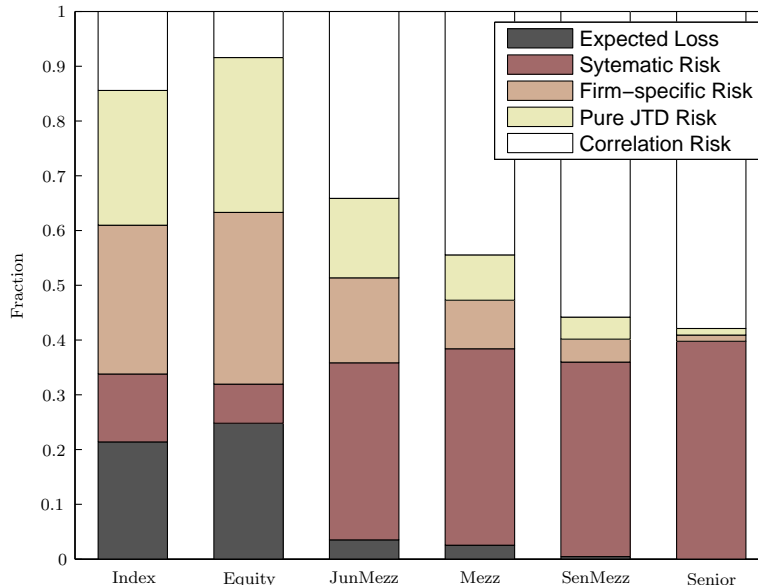


Figure 1: Fractions of index and tranche spreads that are compensation for various sources of risk. Data are for the 5-year CDX North-America Investment-Grade index on November 1, 2006. A precise definition of the risk premia is given in Section 6.2, while Section 7.3 provides intuitive explanations.

sources mentioned above. We see that the equity tranche mostly compensates investors for expected losses, *pure* jump-to-default risk, and the risk of firm-specific changes in credit-worthiness. On the other hand, senior tranches almost exclusively compensate investors for correlation and systematic risk. Such a decomposition of "structured" credit spreads could be applied to a wide range of multi-name credit derivatives, like Nth-to-default baskets, interest- and principal-only tranches, and various types of collateralized debt obligations (CDOs).

In order to estimate a time-series model of tranche pricing that incorporates time variation in conditional default probabilities and time variation in risks and risk premia of the above-mentioned types, we have extended prior work by Duffie and Gârleanu (2001) and Mortensen (2006), that is based on correlated firm-by-firm stochastic intensities of default.² In order to fit the model tractably, we have developed computational techniques, outlined in Eckner (2009), that permits tranche pricing at roughly the same computational speed as for the copula model (Li (2000)), the standard in financial-industry pricing models.

²Standard references on intensity-based models, also called reduced-form models, are Jarrow and Turnbull (1995), Lando (1998), and Duffie and Singleton (1999).

The remainder of this section discusses applications for our work and related literature. We also describe the most common credit derivative contracts, and the data sources used for our analysis. Section 2 presents descriptive statistics of the historical price behavior of some credit derivatives. Section 3 introduces the model for default times and describes the pricing of credit derivatives in this framework. Section 5 examines the model fit to the time series of credit tranche spreads. Section 6 introduces a joint model for physical and risk-neutral default intensities, while Section 7 presents the fitted model and a decomposition of credit tranche spreads and portfolio default compensation. Section 8 concludes.

1.1 Applications

A better understanding of what drives changes in the price of credit risky securities should be of interest to a variety of researchers and practitioners. Asset pricing researchers typically try to link the sensitivity of security prices to certain state variables. The findings in this paper indicate that for some credit derivative contracts, changes in risk premia over time might play a more important role than has traditionally been assigned to them, especially for senior tranche spreads, which heavily depend on perceived tail risk. See also Collin-Dufresne, Goldstein, and Martin (2001), who find that various proposed proxies for changes in default probabilities and recovery rates can explain only about 25 percent of observed credit spread changes.

Investors in single-name and structured credit products would like to better understand the types of risks to which they are exposed to, and to quantify the extent to which they are being compensated for these risks. The decomposition of credit tranche spreads in this paper indicates how a linear combination of credit tranches might allow to isolate the exposure to a certain type of risk, for example *pure* jump-to-default risk, which in the past has carried a rather large risk premium.³ Coval, Jurek, and Stafford (2007) argue that investors in senior CDO tranches underprice systematic risks by solely relying on credit ratings for the purpose of making investment decisions. We find that, at least for corporate credits, investors are reasonably aware, in terms of compensation demanded above and beyond expected losses, about the differences in the risks between structured and single-name credit products, although in our framework we cannot preclude that investors are at least partially oblivious about these differences.

Finally, a better understanding of structured credit products should be of interest to dealers in these securities, since it would allow them to better hedge and quantify their inventory risk. When order flow becomes imbalanced, as in May 2005 and July/August

³The large size of this risk premium has led to the term "credit spread puzzle", see for example Driessen (2005), Berndt, Douglas, Duffie, Ferguson, and Schranz (2005), Chen, Collin-Dufresne, and Goldstein (2006) and Saita (2006). In our framework, the *pure* JTD risk premium accounts for roughly two thirds of the total JTD risk premium, so that the puzzle is somewhat reduced, but remains intact for highly rated issuers.

2007, it is important for liquidity providers to assess whether this imbalance is caused by informed or uninformed traders, so that they can quote more competitive spreads and support an orderly working of the credit markets.

1.2 Related Literature

There exists a vast literature on corporate default risk, initiated by Altman (1968). However, due to computational challenges, most research on the dynamics of physical (or real-world) default intensities (in the following denoted by $\lambda_i^{\mathbb{P}}$) is still quite recent. Duffie, Saita, and Wang (2007) model a firm's default intensity as dependent on a set of firm-specific and macroeconomic covariates, of which distance-to-default, which is a volatility-adjusted measure of leverage, has the most influence. They estimate the time-series dynamics of all covariates and therefore arrive at a fully dynamic model for a firm's default intensity and default time. Duffie, Eckner, Horel, and Saita (2009) extend this model by including a dynamic frailty variable, a time-varying latent factor that affects the default risk of all companies. They find that such a latent factor is important for explaining the tendency of corporate defaults to cluster over time, as in 1989-1990 and 2001-2002, and for obtaining a realistic level of default correlation in intensity-based models.

The recent emergence of a wide array of single-name and structured credit products has made available a tremendous amount of information about investors' risk preferences in the credit markets.⁴ Modeling the dynamics of risk-neutral default intensities (in the following denoted by $\lambda_i^{\mathbb{Q}}$) therefore is currently an extremely active research area. See for example Duffie and Gârleanu (2001), Giesecke and Goldberg (2005), Errais, Giesecke, and Goldberg (2006), Joshi and Stacey (2006), Longstaff and Rajan (2008), Mortensen (2006), Papageorgiou and Sircar (2007), and Eckner (2009). Schneider, Sögner, and Veža (2007) and Feldhütter (2007) examine both the risk-neutral and "physical" dynamics of $\lambda_i^{\mathbb{Q}}$.

Nevertheless, a joint framework for the *multivariate* dynamics of physical and risk-neutral default intensities is still limited. Recent research in this area includes Berndt, Douglas, Duffie, Ferguson, and Schranz (2005), Amato (2005), Amato and Remolona (2005), and Saita (2006). Our paper continues this line of work by incorporating CDX tranche spread data into the analysis, which allows us to better pin down the multivariate dynamics of risk-neutral default intensities and to quantify certain risk premia that are unique to a multivariate setting.

⁴Although corporate bond spreads also contain at least the single-name information, they tend to reflect tax and liquidity effects, see Elton, Gruber, Agrawal, and Mann (2001), Driessen (2005), and Longstaff, Mithal, and Neis (2005). On the other hand, credit default swap spreads are usually regarded as a quite pure measure of perceived default risk. They allow to short credit risk much more easily and cheaper than using corporate bonds, and have standardized maturity dates and credit event triggers, which enhances their liquidity.

1.3 Credit Derivative Contracts

A credit derivative is a contract whose payoff is linked to the creditworthiness of one or more obligations. This section summarizes the features of the three most common types for corporate credit risk.

Credit Default Swaps. By far the most common credit derivative is the credit default swap (CDS). It is an agreement between a protection buyer and a protection seller, whereby the buyer pays a periodic fee in return for a contingent payment by the seller upon a credit event, such as bankruptcy or “failure to pay”, of a reference entity. The contingent payment usually replicates the loss incurred by a creditor of the reference entity in the event of its default. See, for example, Duffie (1999).

Credit Indices. A credit index contract allows an investor to either buy or sell protection on a basket of reference entities, and therefore closely resembles a portfolio of CDS contracts. For example, the CDX.NA.IG (for CDS index, North America, Investment Grade) contract provides equally-weighted default protection on 125 North American investment-grade rated issuers. For a more detailed description see the ‘Credit Derivatives Handbook’ (2006) by Merrill Lynch.

Credit Tranches. A credit tranche allows an investor to gain a specified exposure to the credit risk of the underlying portfolio, while in return receiving periodic coupon payments. Losses due to credit events in the underlying portfolio are allocated first to the lowest tranche, known as the equity tranche, and then to successively prioritized tranches. The risk of a tranche is determined by the attachment point of the tranche, which defines the point at which losses in the underlying portfolio begin to reduce the tranche notional, and the detachment point, which defines the point at which losses in the underlying portfolio reduce the tranche notional to zero. The buyer of protection makes coupon payments on the notional amount of the remaining size of the tranche, which is the initial tranche size less losses due to defaults. This structure is illustrated in Table 1 for the CDX.NA.IG index.

The coupon payments for these three contracts are typically made quarterly, on the IMM dates, which are the 20th of March, June, September, and December, unless this date is a holiday, in which case the payment is made on the next business day. If a contract is entered in between two such dates, the buyer of protection receives from the seller of protection the accrued premium since the last IMM date.

Tranche	Attachment Point	Detachment Point	Quote Convention
Equity	0%	3%	500 bps running + upfront
Junior Mezzanine	3%	7%	all running
Mezzanine	7%	10%	all running
Senior Mezzanine	10%	15%	all running
Senior	15%	30%	all running
Super Senior	30%	100%	all running

Table 1: Tranche structure of the CDX.NA.IG index, which has 125 equally weighted North-American investment-grade issuers in the underlying portfolio.

1.4 Data Sources

Citi and Morgan Stanley provided 5, 7, and 10-year CDX.NA.IG index and tranche bid- and ask-spreads. Markit provided 5, 7, and 10-year CDX.NA.IG tranche mid-market spreads, and 1, 5, 7, and 10-year CDS mid-market spreads for the CDX.NA.IG members.

We used 3-month, 6-month, 9-month, 1-year, 2-year, ..., 10-year US LIBOR swap rates to estimate the discount function $B_t(T)$ at each point in time t . Specifically, for these standard maturities we used swap rates from the Bloomberg system, while for non-standard maturities we used cubic-spline interpolation of implied forward rates to determine the spot rate. Swap rates are widely regarded as the best measure of *funding* costs for banks - the biggest participant in the market for synthetic credit derivatives - over various time horizons.

2 Descriptive Analysis

This section gives a brief overview of historical CDX.NA.IG index and tranche spreads and provides some descriptive statistics. In the subsequent analysis we will frequently refer to the credit market events mentioned here. Unless mentioned otherwise, market prices refer to mid-market prices for the remainder of the paper.

Figure 2 shows the historical spread for the three standard maturities of the CDX.NA.IG index. We see that credit spreads have narrowed considerably between October 2003 and November 2006, although this narrowing was sharply interrupted in May 2005. Corresponding tightenings of spreads affected the tranches making up the index. However, there are some peculiarities in the history of tranche spreads that are worth mentioning in order to interpret the time-series results in Section 5 and 7.

A principal component analysis of the standardized daily tranche spread changes ($\Delta S_j / \sqrt{\text{Var}(\Delta S_j)} : 1 \leq j \leq 5$) with $j = 1$ for the equity tranche, up to $j = 5$ for the

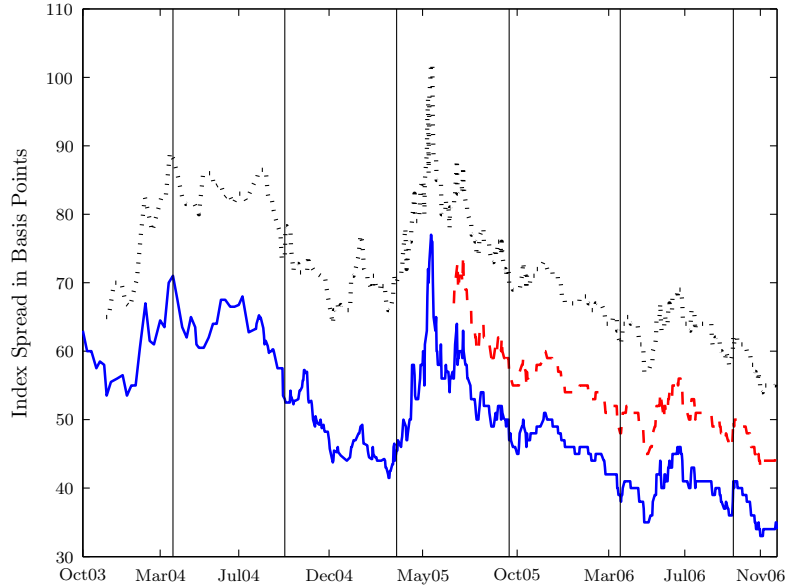


Figure 2: Historical spread of the on-the-run CDX.NA.IG index for maturity equal to 5 years (solid line), 7 years (dashed line) and 10 years (dotted line). The vertical lines represent the bi-annual roll-over dates.

senior tranche, gives the first three principal component weighting vectors

$$PC_1 = \begin{pmatrix} 0.40 \\ 0.48 \\ 0.48 \\ 0.45 \\ 0.42 \end{pmatrix}, PC_2 = \begin{pmatrix} -0.84 \\ 0.05 \\ 0.30 \\ 0.44 \\ -0.07 \end{pmatrix}, PC_3 = \begin{pmatrix} -0.22 \\ -0.32 \\ -0.22 \\ -0.08 \\ 0.89 \end{pmatrix}. \quad (1)$$

The first three principal components explain 76%, 10%, and 8% of the variance of changes in tranche spreads, respectively. As can be seen from (1), the first principal component reflects a market-wide increase of credit risk, the second component reflects an increase of tail risk at moderate loss levels (mezzanine and senior mezzanine tranche), and the third component corresponds to an increase of tail risk at very extreme loss levels (senior tranche). Figure 3 shows the historical evolution of the first three principal components of standardized tranche spread changes between October 2003 and April 2006.⁵ The

⁵We define the evolution of a principal component as the performance of a portfolio of credit tranches with position sizes proportional to the principal component entries, normalized by tranche notional size and tranche spread volatility.

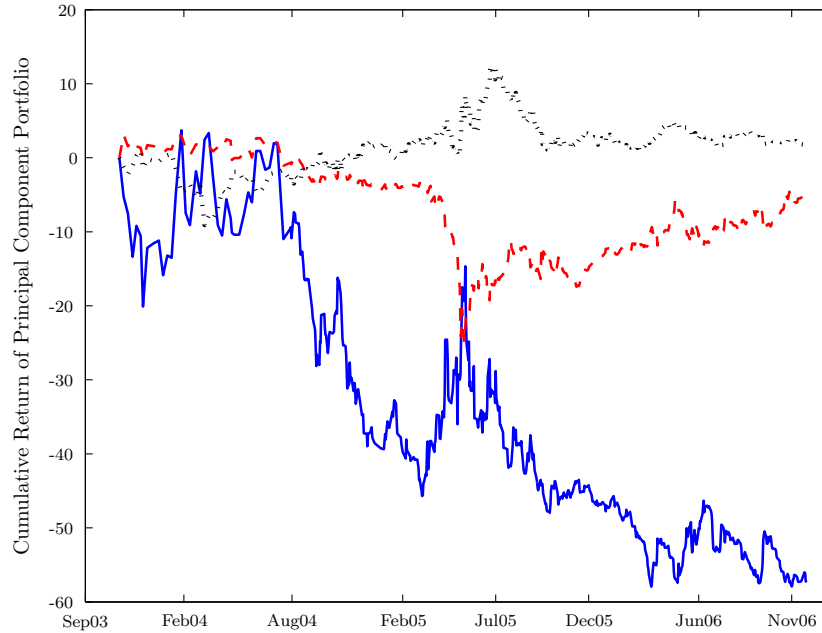


Figure 3: Historical evolution of the first three principal component portfolios (solid, dashed and dotted line, respectively) for the tranche spreads of the 5-year CDX.NA.IG index.

downward trend in the first principal component portfolio simply reflects the fact that credit spreads have narrowed considerably during this time period. The evolution of the second and third component shows that perceived default risk at extreme loss levels increased dramatically in May 2006, while perceived default risk at moderate loss levels apparently collapsed. This shock to the credit market was caused by the same-day downgrades of Ford Motor Company and General Motors Corp. to a sub-investment-grade rating (that is, lower than BBB-), which led to selling of bonds of these firms by investors who are allowed to hold only investment-grade rated securities. At the same time, a partial take-over attempt of General Motors Corp. led to a sharp increase in its share price. The combination of these two events "caused the relationship between prices of certain assets to change in an unexpected way". (See BIS Quarterly Review, June 2005, for a detailed discussion.)

Table 2 provides some summary statistics about the riskiness of reference entities that were part of the CDX.NA.IG index at least once between September 2004 and November 2006.

Rating	Daily Counts	1yr Default Probability	5yr Default Probability	Average 1-yr CDS Spread	Average 5-yr CDS Spread
AAA	2040	0.00%	0.20%	5.3	18.7
AA	1530	0.08%	0.97%	4.6	15.8
A	23066	0.08%	1.37%	7.1	27.5
BBB	33891	0.30%	3.51%	13.2	51.4
BB	7422	1.43%	10.04%	61.4	150.5
B	2040	4.48%	20.89%	187.8	334.2

Table 2: Summary statistics on the default risk of reference entities that were part of the CDX.NA.IG index at least once between September 21, 2004, and November 30, 2006. Historical 1-year and 5-year default probabilities were taken from Moody’s Investor Service, ‘Historical Default Rates of Corporate Bond Issuers, 1920-1999’.

3 Model Setup

This section describes a model for the joint distribution of various obligor default times under a risk-neutral probability measure. The setup is the same as in Eckner (2009), which in turn is similar to the one of Duffie and Gârleanu (2001) and Mortensen (2006).

To this end, we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual conditions.⁶ Up to purely technical conditions,⁷ the absence of arbitrage implies the existence of an equivalent martingale measure \mathbb{Q} , such that the price at time t of a security paying an amount Z at a bounded stopping time $\tau > t$ is given by

$$V_t = E_t^{\mathbb{Q}} \left(e^{-\int_t^\tau r_s ds} Z \right),$$

where r is the short-term interest rate and $E_t^{\mathbb{Q}}$ denotes expectation under \mathbb{Q} conditional on all available information up to time t .

Under the equivalent martingale measure \mathbb{Q} , for each individual firm i , a default time τ_i is modeled using Cox processes, also known as doubly stochastic Poisson processes.⁸ Specifically, the default intensity of obligor i is a non-negative real-valued progressively measurable stochastic process, which will be defined below. Conditional on the intensity path $\{\lambda_{it}^{\mathbb{Q}} : t \geq 0\}$, the default time τ_i is taken to be the first jump time of an inhomogeneous Poisson process with intensity $\lambda_i^{\mathbb{Q}}$. In particular, the default times of any set of firms are conditionally independent given the intensity paths, so that correlation of default intensities is the only mechanism by which correlation of default times can arise.

⁶For a precise mathematical definition not offered here, see Karatzas and Shreve (2004) and Protter (2005).

⁷See Harrison and Kreps (1979), Harrison and Pliska (1981), and Delbaen and Schachermayer (1999).

⁸See for example Lando (1998) and Duffie and Singleton (2003).

For $t > s$, risk-neutral survival probabilities can be calculated via

$$\mathbb{Q}(\tau_i > t | \mathcal{F}_s) = E_s^{\mathbb{Q}}(\mathbb{Q}(\tau_i > t | \{\lambda_{it}^{\mathbb{Q}} : t \geq 0\} \cup \mathcal{F}_s)) = \mathbf{1}_{\{\tau_i > s\}} E_s^{\mathbb{Q}}\left(e^{-\int_s^t \lambda_{iu}^{\mathbb{Q}} du}\right), \quad (2)$$

where the expectation is taken over the distribution of possible intensity paths. The large and flexible class of affine processes allows one to calculate (2) either explicitly or numerically quite efficiently. See Duffie, Pan, and Singleton (2000), and Duffie, Filipović, and Schachermayer (2003) for a more general discussion of affine processes.

Due to their computational tractability, we use so-called *basic* Affine Jump Diffusions (bAJD) as the building block for the default intensity model. Specifically, we call a stochastic process Z a basic AJD under \mathbb{Q} if

$$dZ_t = \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - Z_t) dt + \sigma \sqrt{Z_t} dB_t^{\mathbb{Q}} + dJ_t^{\mathbb{Q}}, \quad Z_0 \geq 0, \quad (3)$$

where under \mathbb{Q} , $(B_t^{\mathbb{Q}})_{t \geq 0}$ is a standard Brownian motion, and $(J_t^{\mathbb{Q}})_{t \geq 0}$ is an independent compound Poisson process with constant jump intensity $l^{\mathbb{Q}}$ and exponentially distributed jumps with mean $\mu^{\mathbb{Q}}$. For the process to be well defined, we require that $\kappa^{\mathbb{Q}}\theta^{\mathbb{Q}} \geq 0$ and $\mu^{\mathbb{Q}} \geq 0$.

3.1 Risk-Neutral Default Intensities

We now make precise the multivariate model of default times for m obligors. For $1 \leq i \leq m$, the risk-neutral default intensity of obligor i is

$$\lambda_{it}^{\mathbb{Q}} = X_{it} + a_i Y_t, \quad (4)$$

with idiosyncratic component X_i and systematic component Y . Under \mathbb{Q} , X_1, \dots, X_m and Y are independent basic AJDs with

$$dX_{it} = \kappa_i^{\mathbb{Q}}(\theta_i^{\mathbb{Q}} - X_{it})dt + \sigma_i \sqrt{X_{it}} dB_t^{\mathbb{Q},(i)} + dJ_t^{\mathbb{Q},(i)} \quad (5)$$

$$dY_t = \kappa_Y^{\mathbb{Q}}(\theta_Y^{\mathbb{Q}} - Y_t)dt + \sigma_Y \sqrt{Y_t} dB_t^{\mathbb{Q},(Y)} + dJ_t^{\mathbb{Q},(Y)}. \quad (6)$$

Here, $J^{\mathbb{Q},(Y)}$ and $J^{\mathbb{Q},(i)}$ have jump intensities $l_Y^{\mathbb{Q}}$ and $l_i^{\mathbb{Q}}$, and jump size means $\mu_Y^{\mathbb{Q}}$ and $\mu_i^{\mathbb{Q}}$, respectively. Hence, jumps in default intensities can either be firm-specific or market-wide. Duffie and Gârleanu (2001) and Mortensen (2006) found the latter type of jumps to be crucial for explaining the spreads of senior CDO tranches, which are heavily exposed to tail risk events. Schneider, Sögner, and Veža (2007) examined the time series of 282 credit default swap spreads and found evidence for mainly positive jumps in default intensities.

3.1.1 Parameter Restrictions

This section discusses restrictions on the parameters in (5) and (6) that (i) make the model identifiable and (ii) reduce, for parsimony, the number of free parameters. The constraints are the same as in Eckner (2009) and similar to those of Duffie and Gârleanu (2001) and Mortensen (2006).

Model Identifiability. The restriction

$$\frac{1}{m} \sum_{i=1}^m a_i = 1$$

is imposed to ensure identifiability of the model.⁹

Parsimony. Our model specification is relatively general with $5m + 5$ default intensity parameters and 2 liquidity parameters, as well as $m + 1$ initial values for the factors. Since we are especially interested in the economic interpretation of the parameters, we favor a parsimonious model which is nevertheless flexible enough to closely fit tranches spreads. First, we take the common factor loading a_i of each obligor i to be equal to the obligor's 5-year CDS spread divided by the average 5-year CDS spread of the current credit index members, that is

$$a_i = \frac{c_{i,t,5}^{\text{cds}}}{\text{Avg}(c_{i,t,5}^{\text{cds}})}, \quad (7)$$

where $c_{i,t,5}^{\text{cds}}$ denotes the 5-year CDS spread at time t for the i -th reference entity. Moreover, we impose the parameter constraints

$$\kappa_i^{\mathbb{Q}} = \kappa_Y^{\mathbb{Q}} \equiv \kappa^{\mathbb{Q}}, \quad (8)$$

$$\sigma_i = \sqrt{a_i} \sigma_Y \equiv \sqrt{a_i} \sigma, \quad (9)$$

$$\mu_i^{\mathbb{Q}} = a_i \mu_Y^{\mathbb{Q}} \equiv a_i \mu^{\mathbb{Q}}, \quad (10)$$

$$\omega_1 = \frac{l_Y^{\mathbb{Q}}}{l_i^{\mathbb{Q}} + l_Y^{\mathbb{Q}}}, \quad (11)$$

$$\omega_2 = \frac{a_i \theta_Y^{\mathbb{Q}}}{a_i \theta_Y^{\mathbb{Q}} + \theta_i^{\mathbb{Q}}}, \quad (12)$$

which reduces the number of free parameters to just seven. Feldhütter (2007) examines to what extent (7)-(12) are empirically supported by CDS data for firms in the CDX.NA.IG index, and finds these assumptions in general to be fairly reasonable. See also Eckner (2009).

The constraints (7)-(12) imply that $\lambda_i^{\mathbb{Q}}$ is also a basic AJD, which is not generally the case for the sum of two basic AJDs, see Duffie and Gârleanu (2001), Proposition 1. Specifically,

$$d\lambda_{it}^{\mathbb{Q}} = \kappa \left((\theta_i^{\mathbb{Q}} + a_i \theta_Y^{\mathbb{Q}}) - \lambda_{it} \right) dt + \sqrt{a_i} \sigma \sqrt{\lambda_{it}^{\mathbb{Q}}} d\tilde{B}_t^{\mathbb{Q},(i)} + d\tilde{J}_t^{\mathbb{Q},(i)},$$

⁹If all factor loadings a_i are replaced by ca_i for some positive constant c , then replacing the parameters $(Y_0, \kappa_Y^{\mathbb{Q}}, \theta_Y^{\mathbb{Q}}, \sigma_Y, l_Y^{\mathbb{Q}}, \mu_Y^{\mathbb{Q}})$ with $(Y_0/c, \kappa_Y^{\mathbb{Q}}, \theta_Y^{\mathbb{Q}}/c, \sigma_Y/\sqrt{c}, l_Y^{\mathbb{Q}}, \mu_Y^{\mathbb{Q}}/c)$ leaves the dynamics of $a_i Y$ (and therefore also the joint dynamics of $\lambda_i^{\mathbb{Q}}$) unchanged.

or in short-hand

$$\begin{aligned}\lambda_i^{\mathbb{Q}} &= \text{bAJD}(\lambda_{i,0}^{\mathbb{Q}}, \kappa^{\mathbb{Q}}, \theta_i^{\mathbb{Q}} + a_i \theta_Y^{\mathbb{Q}}, \sqrt{a_i} \sigma_Y, l_i^{\mathbb{Q}} + l_Y^{\mathbb{Q}}, \mu_i^{\mathbb{Q}}) = \\ &= \text{bAJD}(\lambda_{i,0}^{\mathbb{Q}}, \kappa^{\mathbb{Q}}, \tilde{\theta}_i^{\mathbb{Q}}, \tilde{\sigma}_i, \tilde{l}^{\mathbb{Q}}, \mu_i^{\mathbb{Q}}),\end{aligned}$$

where $\tilde{\theta}_i^{\mathbb{Q}} \equiv \theta_i^{\mathbb{Q}} + a_i \theta_Y^{\mathbb{Q}}$ and $\tilde{l}^{\mathbb{Q}} \equiv l_i^{\mathbb{Q}} + l_Y^{\mathbb{Q}}$.

It is easy to show that $\theta_Y = \omega_2 \text{Avg}(\tilde{\theta}_i^{\mathbb{Q}}) \equiv \omega_2 \tilde{\theta}_{\text{Avg}}^{\mathbb{Q}}$ and that $\tilde{\theta}_i^{\mathbb{Q}} = a_i \tilde{\theta}_{\text{Avg}}^{\mathbb{Q}}$ for each i , so that we can characterize the risk-neutral model of joint default times by the seven parameters

$$\Theta^{\mathbb{Q}} = \left\{ \kappa^{\mathbb{Q}}, \tilde{\theta}_{\text{Avg}}^{\mathbb{Q}}, \sigma, \tilde{l}^{\mathbb{Q}}, \mu^{\mathbb{Q}}, \omega_1, \omega_2 \right\}$$

and the $m+1$ initial values of the factors.¹⁰ Even though the constraints (7)-(12) greatly simplify the model setup, a model without these constraints would be just as tractable, since the computational techniques described in Eckner (2009) could still be applied.

4 Pricing and Model Calibration

After specifying the multivariate default intensity dynamics, we turn to the pricing of various credit derivatives. The general procedure for pricing credit derivatives is setting the value of the fixed leg (the market value of the payments made by the buyer of protection) equal to the value of the protection leg (the market value of the payments made by the seller of protection) and to solve for the fair credit spread. Since the pricing of credit risky securities in this framework is by now standard, we keep this section fairly short and refer the interested reader to Mortensen (2006) and Eckner (2009) for a detailed description.

For the purpose of pricing credit risky securities, we adopt the widely used assumption:¹¹

Assumption 1 *Under the risk-neutral probability measure \mathbb{Q} ,*

- (i) *Default intensities and interest rates are independent.*
- (ii) *Recovery rates are independent of default intensities.*

As shown by Mortensen (2006) and Eckner (2009), model-implied CDS and credit index spreads can be calculated explicitly under assumption 1, while model-implied

¹⁰As in Eckner (2009), we also incorporated two liquidity premia parameters for CDS and credit tranche contracts. However, they are of minor importance for the subsequent analysis.

¹¹See Eckner (2009) for a discussion of these assumptions.

tranche spreads can be calculated in a computationally efficient manner. For completeness, we illustrate the pricing of the fixed leg of a CDS contract with set of coupon payment dates $\{t_l : 1 \leq l \leq n\}$. At a time s and under Assumption 1, the market value the fixed equals

$$V_t^{\text{cds,Fixed}}(s) = N_i c_i^{\text{cds}} \sum_{\{l:t_l>s\}} B_s(t_l) \frac{(t_l - t_{l-1})}{360} \mathbb{Q}_s(\tau_i > t_l) - N_i c_i^{\text{cds}} \frac{s - \max(t_l : t_l \leq s)}{360}, \quad (13)$$

using an Actual/360 day-count convention, where N_i is the notional of the CDS contract, c_i^{cds} the coupon, and $B_t(T)$ is the discount function at time t for a unit payoff at maturity $T \geq t$. The first term in (13) is the present value of future coupon payments, while the second term reflects the accrued premium since the most recent coupon payment date. In the affine two-factor model of Section 3.1, the survival probabilities $\mathbb{Q}_s(\tau_i > t_l)$ are known explicitly via the moment generating function of a basic AJD. See Duffie and Gârleanu (2001), and Eckner (2009) for details. Hence, the market value of the fixed leg (13) can be calculated explicitly in this framework. Again, see Mortensen (2006) and Eckner (2009) for the details on valuing the default leg of a CDS contract, and on valuing credit indices and tranches in this framework.

4.1 Market Frictions

When pricing credit derivatives one must account for various market frictions, as well as for certain trading conventions in this market. First, credit indices and credit tranches recognize only bankruptcy and “failure to pay” as credit events, whereas CDS contracts usually also include certain forms of restructuring. Second, index arbitrage traders are the medium by which the level of a CDS index is kept in line with so-called *intrinsic*s, which is the fair index level implied by individual CDS spreads. However, because of execution risk and the cost of paying the bid-offer spread when trading individual credit default swaps, index arbitrage traders only become active when the index level differs by at least a couple of basis points from the *intrinsic*s. To account for these effects, we incorporated a time-varying index-CDS basis and index-tranche basis into the model, see Eckner (2009) for details.

4.2 Model Calibration to CDS and Tranche Spreads

This section links model-implied and mid-market CDS, credit index, and credit tranche spreads. To this end, let $c_{j,t,M}^{\text{tr}}(S)$ for $S \in \{\text{MI}, \text{MK}\}$ denote the spread at time t of the j -th tranche with maturity M (usually 5, 7 or 10 years) as implied by the model (MI), and as reported by Markit (MK). Similarly, let $c_{i,t,M}^{\text{cds}}(\text{MK})$ denote the spread at time t

of the i -th CDS with maturity M , as reported by Markit. Finally, let $c_{t,M}^{\text{idx}}(\text{C})$ be the M -year index spread at time t as reported by Citi.

Model-implied and market-observed spreads satisfy the measurement equations

$$\begin{aligned} c_{j,t,M}^{\text{tr}}(\text{MK}) &= c_{j,t,M}^{\text{tr}}(\text{MI}) + \varepsilon_{j,t,M}^{\text{tr}}(\text{MK}) \\ c_{i,t,M}^{\text{cds}}(\text{MK}) &= c_{i,t,M}^{\text{cds}}(\text{MI}) + \varepsilon_{i,t,M}^{\text{cds}}(\text{MK}) \\ c_{t,M}^{\text{idx}}(\text{C}) &= c_{t,M}^{\text{idx}}(\text{MI}) + \varepsilon_{t,M}^{\text{idx}}(\text{C}), \end{aligned} \quad (14)$$

where the measurement errors $\varepsilon_{j,t,M}^{\text{tr}}(\text{MK})$, $\varepsilon_{i,t,M}^{\text{cds}}(\text{MK})$, and $c_{t,M}^{\text{idx}}(\text{C})$ are independent under \mathbb{P} across time, tranche/CDS, maturity and data source, and distributed as

$$\begin{aligned} \varepsilon_{j,t,M}^{\text{tr}}(\text{MK}) &\sim N(0, c_{j,t,M}^{\text{tr}}(\text{MK})^2 \sigma_{\text{tr}}^2) \\ \varepsilon_{i,t,M}^{\text{cds}}(\text{MK}) &\sim N(0, c_{i,t,M}^{\text{tr}}(\text{MK})^2 \sigma_{\text{cds}}^2) \\ \varepsilon_{t,M}^{\text{idx}}(\text{C}) &\sim N(0, c_{t,M}^{\text{idx}}(\text{C})^2 \sigma_{\text{idx}}^2). \end{aligned}$$

Under these assumptions, the log-likelihood function is of the form

$$\log L_{\Theta^{\mathbb{Q}}}^{\mathcal{M}}(\text{CDS}, \text{IDX}, \text{TR}) = \alpha + \beta \cdot \text{RMSE}_{\text{tr}}^2 + \gamma \cdot \text{RMSE}_{\text{cds}}^2 + \delta \cdot \text{RMSE}_{\text{idx}}^2 \quad (15)$$

for some constants $\alpha, \beta, \gamma, \delta$ with $\beta, \gamma, \delta < 0$, where \mathcal{M} denotes the set of maturities under consideration, and CDS , IDX and TR denote the panel data of observed CDS, credit index, and tranche spreads, respectively. Furthermore, RMSE^{tr} is the relative root mean square tranche pricing error defined by

$$\text{RMSE}^{\text{tr}}(\Theta^{\mathbb{Q}}, \mathcal{M}) = \sum_{t=1}^T \text{RMSE}_t^{\text{tr}}(\Theta^{\mathbb{Q}}, \mathcal{M}) \quad (16)$$

$$\text{RMSE}_t^{\text{tr}}(\Theta^{\mathbb{Q}}, \mathcal{M}) = \sqrt{\frac{1}{J} \frac{1}{|\mathcal{M}|} \sum_{j=1}^J \sum_{M \in \mathcal{M}} \left(\frac{c_{j,t,M}^{\text{tr}}(\text{MI}) - c_{j,t,M}^{\text{tr}}(\text{MK})}{c_{j,t,M}^{\text{tr}}(\text{MK})} \right)^2}, \quad (17)$$

RMSE^{cds} is the relative root mean square CDS pricing error

$$\text{RMSE}^{\text{cds}}(\Theta^{\mathbb{Q}}, \mathcal{M}) = \sum_{t=1}^T \text{RMSE}_t^{\text{cds}}(\Theta^{\mathbb{Q}}, \mathcal{M}) \quad (18)$$

$$\text{RMSE}_t^{\text{cds}}(\Theta^{\mathbb{Q}}, \mathcal{M}) = \sqrt{\frac{1}{m} \frac{1}{|\mathcal{M}|} \sum_{i=1}^m \sum_{M \in \mathcal{M}} \left(\frac{c_{i,t,M}^{\text{cds}}(\text{MI}) - c_{i,t,M}^{\text{cds}}(\text{MK})}{c_{i,t,M}^{\text{cds}}(\text{MK})} \right)^2},$$

and RMSE^{idx} is the relative root mean square credit index pricing error

$$\text{RMSE}^{\text{idx}}(\Theta^{\mathbb{Q}}, \mathcal{M}) = \sum_{t=1}^T \text{RMSE}_t^{\text{idx}}(\Theta^{\mathbb{Q}}, \mathcal{M}) \quad (19)$$

$$\text{RMSE}_t^{\text{idx}}(\Theta^{\mathbb{Q}}, \mathcal{M}) = \sqrt{\frac{1}{|\mathcal{M}|} \sum_{M \in \mathcal{M}} \left(\frac{c_{t,M}^{\text{idx}}(\text{MI}) - c_{t,M}^{\text{idx}}(\text{C})}{c_{t,M}^{\text{idx}}(\text{C})} \right)^2}.$$

Maximizing the log likelihood (15) is therefore equivalent to minimizing a weighted sum of squares of RMSE^{tr} , RMSE^{cds} , and RMSE^{idx} . The fitting criterion is thus of the form

$$C(\Theta^{\mathbb{Q}}) = \omega_{\text{tr}}\text{RMSE}^{\text{tr}}(\Theta^{\mathbb{Q}}, \mathcal{M})^2 + \omega_{\text{cds}}\text{RMSE}^{\text{cds}}(\Theta^{\mathbb{Q}}, \mathcal{M})^2 + \omega_{\text{idx}}\text{RMSE}^{\text{idx}}(\Theta^{\mathbb{Q}}, \mathcal{M})^2, \quad (20)$$

where the weights ω_{tr} , ω_{cds} and ω_{idx} are inversely related to the noisiness of the reported market data.

Remark 1 Rather than making strong independence assumptions regarding the measurement errors in (14), one can alternatively directly define (20) as the fitting criterion. The likelihood framework will however be useful for parameter inference in Section 6.

The following algorithm was used for minimizing (20), that is, for fitting the basic AJD model of Section 3.1 to market-observed CDS, credit index, and credit tranche spreads:

Algorithm 1

1. For fixed parameter vector Θ and initial systematic intensity Y_0 , individual CDS spreads are calibrated by varying the initial intensities λ_{i0} for $1 \leq i \leq m$ subject to the constraint $\lambda_{i0} \geq a_i Y_0$, and minimizing the fitting criterion (19).
2. Solve for the index-CDS and index-tranche basis, see Eckner (2009).
3. Vary the parameter vector Θ and the initial systematic intensity Y_0 to minimize the criterion function (20). At each revision of Θ or Y_0 , Step 1 and 2 are repeated.

Step 3 was implemented by fitting each parameter separately and iterating over the set of parameters. Convergence typically occurred after 20 to 30 iterations.

5 Results – Time Series Analysis

Mortensen (2006) and Eckner (2009) examine the fit of a basic AJD model to CDX tranche spreads at a *fixed* point in time. This section examines the fit of such a model, at a weekly frequency, to 5-year CDS, index and tranche spreads between September 2004 and November 2006.

We first examined a model with fixed parameters, so that only the idiosyncratic factors X_{it} and the systematic factor Y_t can vary over time, but for example not their jump intensity or volatility. As expected, the model fit to the time series of tranches spreads turned out to be poor, with relative tranche pricing errors frequently exceeding

50%. However, expecting a model with fixed parameters to fit the time series of tranche spreads over a multi-year time-horizon is probably quite unrealistic, simply because investors' risk aversion often changes dramatically over time. See also Feldhütter (2007), who fitted a similar constant parameter model to six months of data, and found that time-variation in the senior mezzanine and senior tranche spreads cannot be captured by such a model.

In this section, we therefore examine a model with time-varying parameters. By this we mean fitting a separate model for each point in time t , so that each date has its own set of risk-neutral default intensity parameters $\Theta_t^{\mathbb{Q}}$. Hence, parameters are allowed to change arbitrarily from one day to the next one without imposing any time series constraints, so that for example the realized volatility of the common factor Y need not necessarily be consistent with the model parameter σ_Y . This section therefore merely examines the potential of the basic AJD model from Section 3 to fit market-observed tranche spreads at different points in time. This procedure is a relatively weak test of model specification, but one that is common in the financial industry.

5.1 Empirical Analysis

For this section, we replace the definition of the relative RMSE (17) by the following robust version

$$\widetilde{\text{RMSE}}_t^{\text{tr}}(\Theta_t^{\mathbb{Q}}, \mathcal{M}) = \sqrt{\frac{1}{J} \frac{1}{|\mathcal{M}|} \sum_{j=1}^J \sum_{M \in \mathcal{M}} (\text{RE}_{j,t,M})^2}, \quad (21)$$

where

$$\begin{aligned} \text{RE}_{j,t,M}^2 &= \min \left[\left(\frac{c_{j,t,M}^{\text{tr}}(\text{MI}) - c_{j,t,M}^{\text{tr}}(\text{MK})}{c_{j,t,M}^{\text{tr}}(\text{MK})} \right)^2, \left(\frac{c_{j,t,M}^{\text{tr}}(\text{MI}) - c_{j,t,M}^{\text{tr}}(\text{MS})}{c_{j,t,M}^{\text{tr}}(\text{MS})} \right)^2 \right] \\ &\quad \times \left(\mathbf{1}_{\{c_{j,t,M}^{\text{tr}}(\text{MI}) < \min(c_{j,t,M}^{\text{tr}}(\text{MS}), c_{j,t,M}^{\text{tr}}(\text{MK}))\}} \right. \\ &\quad \left. + \mathbf{1}_{\{c_{j,t,M}^{\text{tr}}(\text{MI}) > \max(c_{j,t,M}^{\text{tr}}(\text{MS}), c_{j,t,M}^{\text{tr}}(\text{MK}))\}} \right), \end{aligned} \quad (22)$$

and where $c_{j,t,M}^{\text{tr}}(\text{MI})$, $c_{j,t,M}^{\text{tr}}(\text{MK})$, and $c_{j,t,M}^{\text{tr}}(\text{MS})$ denote for time t the spread of the j -th tranche with maturity M , as implied by the model (MI), reported by Markit (MK) and reported by Morgan Stanley (MS), respectively. In particular, the relative pricing error (22) is zero if the model-implied tranche spread lies in between the market mid-prices reported by Markit and Morgan Stanley. The fitting criterion (21) mitigates the effect of a few outliers in the data that would otherwise cloud the results of the subsequent analysis.

Figure 4 shows the time series of robust relative tranche pricing errors (22) for the basic AJD model of Section 3.1, when fitted to the 5-year CDX.NA.IG index between

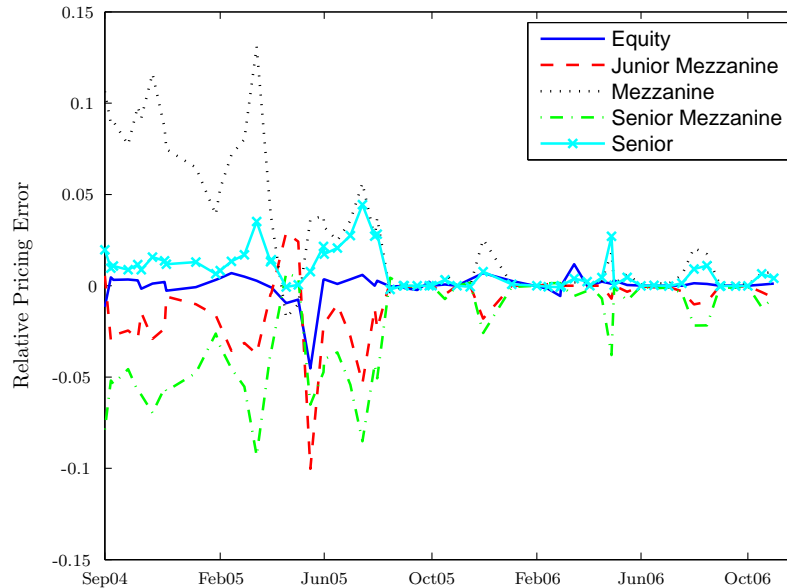


Figure 4: Robust relative tranche pricing errors (22) for the fitted basic AJD model of Section 3.1. Positive values correspond to a model-implied spread above the market-observed spread, and vice versa.

September 2004 and November 2006.¹² First, we see a pronounced downward trend over time in the relative magnitude of pricing errors.¹³ Since we are measuring pricing errors in *relative* terms, this pattern does not simply reflect the fact that credit spreads have narrowed considerably between September 2004 and November 2006. The reduction of *absolute* tranche pricing errors in fact has been even more pronounced. Consistent with Longstaff and Rajan (2008), we find that the largest pricing errors occurred shortly after the introduction of the CDX index and tranche contracts, but that these discrepancies largely disappeared within a couple of months, potentially reflecting an evolution towards a more efficient market for structured credit derivatives, or at least more homogeneity in the models applied in industry practice.¹⁴

¹²On dates for which either not all CDS quotes were available or for which we were unable to match all index members with the individual company identifiers, we calculated the portfolio loss distribution by excluding these companies, and by scaling up the index exposure with respect to the remaining firms in order to keep the portfolio notional constant.

¹³Using (17) as of the relative RMSE, instead of robust version (21), would have caused a few one-day spikes in the tranche pricing errors to appear in Figure 4. An analysis of the tranche spreads as reported by Markit and Morgan Stanley revealed, that in most of these cases, Morgan Stanley's reported bid-ask range of the tranche spread did not include the market-mid price reported by Markit, and vice versa.

¹⁴However, when fitting the whole term-structure of 5, 7, and 10-year tranche spreads instead of just

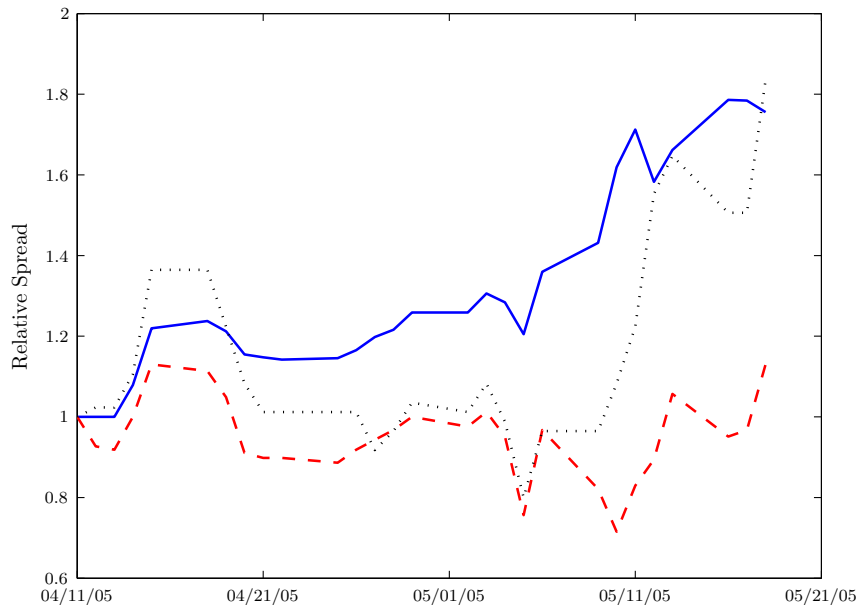


Figure 5: Normalized credit spread of the equity (solid line), mezzanine (dashed line) and senior tranche (dotted line) between April 11, 2005, and May 18, 2005, for the 5-year CDX.NA.IG index.

The downward trend in relative tranche pricing errors was interrupted by a sharp increase around May 2005, an episode in the credit market that is now commonly referred to as the "GM downgrade", or the "correlation crunch", see BIS Quarterly Review, June 2005. Figure 5 shows the normalized tranche spread for the equity,¹⁵ mezzanine and senior tranche of the 5-year CDX.NA.IG index between April 11, 2005, and May 18, 2006. We see that, over a period of a few weeks, the spread of the equity and senior tranche increased by about 80%, while the spread of the mezzanine tranche hardly changed at all. Since the mezzanine tranche is located in between the equity and senior tranche in terms of priority of payments, such a behavior of tranche spreads can be explained only by an extreme change in perceived default correlation. Figure 4 shows that this change in default correlation cannot be captured by the basic AJD model of Section 3.1, but that the equity, junior mezzanine and senior mezzanine tranche were trading 5-10% above the model-implied spread on May 16, 2005.

5-year spreads, pricing discrepancies are still quite large, although there seems to be a downward trend present.

¹⁵In case of the equity tranche we used the normalized up-front payment.

5.2 Parameter Stability over Time

Table 3 reports the fitted model parameters σ_Y , $\kappa^{\mathbb{Q}}\tilde{\theta}_{\text{Avg}}^{\mathbb{Q}} + \tilde{l}^{\mathbb{Q}}\mu^{\mathbb{Q}}$, and ω_1 at a monthly frequency between September 2004 and November 2006.¹⁶ The first column shows a pronounced downward trend in the volatility of default intensities, and therefore also of CDS spread volatilities, since both quantities are closely related. The second column indicates that optimism about the future credit market environment increased between September 2004 and November 2006. The last column, which shows the ratio of the systematic to total jump intensity, indicates that perceived systematic jump risk declined relative to firm-specific jump risk. In summary, the parameter time series in Table 3 show a significant reduction in perceived market-wide credit risk between September 2004 and November 2006. We also see that the risk-neutral default intensity parameter vector $\Theta_t^{\mathbb{Q}}$ is relatively stable over time, especially for periods during which tranche spreads did not change much, which is important for the model's applicability to hedging. See Eckner (2009) for a discussion of hedging tranches against various risk factors in the credit market, when using the basic AJD framework of Section 3.1.

¹⁶The time series of the other model parameters are qualitatively similar and available upon request. See Eckner (2009) for a discussion and interpretation of these parameters.

Date	σ_Y	$(\kappa^{\mathbb{Q}} \tilde{\theta}_{\text{Avg}}^{\mathbb{Q}} + \tilde{l}^{\mathbb{Q}} \mu^{\mathbb{Q}}) \times 10^4$	ω_1
9/21/2004	0.122	37.1	0.43
11/2/2004	0.134	39.9	0.42
12/1/2004	0.122	34.2	0.35
1/4/2005	0.106	24.6	0.36
2/1/2005	0.131	37.9	0.42
3/1/2005	0.128	39.1	0.45
4/1/2005	0.122	44.1	0.48
5/2/2005	0.105	52.2	0.58
6/1/2005	0.104	44.7	0.36
7/1/2005	0.081	35.6	0.48
8/1/2005	0.084	24.3	0.48
9/1/2005	0.093	24.3	0.59
10/3/2005	0.091	18.6	0.36
11/1/2005	0.094	26.2	0.27
12/1/2005	0.109	35.3	0.24
1/3/2006	0.106	26.6	0.29
2/1/2006	0.093	21.9	0.28
3/1/2006	0.083	18.2	0.36
4/3/2006	0.072	27.8	0.21
5/1/2006	0.083	26.6	0.20
6/1/2006	0.093	33.9	0.25
7/3/2006	0.091	30.9	0.15
8/1/2006	0.098	31.2	0.20
9/1/2006	0.088	26.5	0.20
10/2/2006	0.084	24.0	0.28
11/1/2006	0.083	22.5	0.19

Table 3: Time series of fitted model parameters at a monthly frequency between September 2004 and November 2006. σ_Y is the volatility of the systematic risk factor, $\kappa^{\mathbb{Q}}$ the mean reversion speed of systematic and idiosyncratic risk factors, $\tilde{\theta}_{\text{Avg}}^{\mathbb{Q}}$ the average mean reversion level of default intensities, $\tilde{l}^{\mathbb{Q}}$ the jump intensity and $\mu^{\mathbb{Q}}$ the average expected jump size of default intensities, and ω_1 the fraction of jumps that are systematic.

6 Joint Model for $\lambda^{\mathbb{P}}$ and $\lambda^{\mathbb{Q}}$

This section describes a joint model for physical and risk-neutral default intensities, which allows one to quantify the sizes of various types of risk premia in the credit market. We start with a couple of remarks about empirical features that such a joint model should be able to capture.

In general, there are two types of default intensities that are of interest for a given company, namely its physical (or real-world) default intensity (that under \mathbb{P} is denoted by $\lambda_i^{\mathbb{P}}$) and its risk-neutral default intensity (that under \mathbb{Q} is denoted by $\lambda_i^{\mathbb{Q}}$). The quantity

$$\frac{\lambda_{it}^{\mathbb{Q}}}{\lambda_{it}^{\mathbb{P}}} - 1$$

is commonly referred to as *the* jump-to-default (JTD) risk premium for company i at time t , and can be seen as instantaneous compensation to investors for being exposed to the risk of instantaneous default of company i . Under the conditional diversification hypothesis (see Jarrow, Lando, and Yu (2005)), the jump-to-default risk premium is equal to zero, since JTD risk can be diversified away. However, empirical studies have found the average value of $\lambda_{it}^{\mathbb{Q}}/\lambda_{it}^{\mathbb{P}} - 1$ to be somewhere in the range between 1.5 and 4 for corporate bonds and credit-default-swaps of BBB rated US issuers, although this value can fluctuate strongly over time. See for example Berndt, Douglas, Duffie, Ferguson, and Schranz (2005), Amato and Remolona (2005) and Saita (2006). Amato (2005) finds the JTD risk premium to be inversely related to credit ratings, and to be as high as 625 for AAA and as low as 2.2 for BB-rated debt. These findings indicate that the liquid universe of credit risky securities is probably not large enough to diversify away short-term default risk and that investors therefore demand *instantaneous* compensation above and beyond compensation for *instantaneous* expected losses when holding a portfolio of corporate credits. In addition, borrowing constraints are likely to be at least partially responsible for the dramatic variation of the JTD risk premium across rating categories. For example, an investor subject to borrowing constraints might be able to reach a certain target expected return only by investing in high-yielding assets, but not by taking a leveraged position in lower-yielding assets.¹⁷

A second type of risk premia that is deemed to be important in the credit market is compensation for mark-to-market risk, which is the uncertainty regarding the future evolution of risk factors in general, and default intensities in particular. See for example Berndt, Douglas, Duffie, Ferguson, and Schranz (2005) and Saita (2006).

¹⁷A similar explanation has been offered for the historical risk-adjusted outperformance of low-beta over high-beta stocks. See Black, Jensen, and Scholes (1972) and Black (1993).

6.1 Joint Model Specification

This section makes precise the joint model for physical and risk-neutral default intensities. We again use an affine two-factor model of the form

$$\begin{aligned}\lambda_{it}^{\mathbb{P}} &= b_{it}^{\mathbb{P}} X_{it} + a_{it}^{\mathbb{P}} Y_t, \\ \lambda_{it}^{\mathbb{Q}} &= X_{it} + a_{it}^{\mathbb{Q}} Y_t,\end{aligned}\tag{23}$$

for $1 \leq i \leq m$, where X_i and Y are independent basic AJDs under \mathbb{P} and under \mathbb{Q} . We use time-varying model parameters $\Theta_t^{\mathbb{Q}}$, but for notational simplicity suppress this time-dependence in most equations.¹⁸ The dynamics of X_i are given by

$$\begin{aligned}dX_{it} &= \kappa_i^{\mathbb{P}} (\theta_i^{\mathbb{P}} - X_{it}) dt + \sigma_i \sqrt{X_{it}} dB_t^{\mathbb{P},(i)} + dJ_t^{\mathbb{P},(i)} \\ dX_{it} &= \kappa_i^{\mathbb{Q}} (\theta_i^{\mathbb{Q}} - X_{it}) dt + \sigma_i \sqrt{X_{it}} dB_t^{\mathbb{Q},(i)} + dJ_t^{\mathbb{Q},(i)}\end{aligned}\tag{24}$$

for $1 \leq i \leq m$. Under \mathbb{P} , $B^{\mathbb{P},(i)}$ are independent Brownian motions and $J^{\mathbb{P},(i)}$ independent compound Poisson processes. Under \mathbb{Q} , $B^{\mathbb{Q},(i)}$ are independent Brownian motions and $J^{\mathbb{Q},(i)}$ independent compound Poisson processes.

Similarly, the dynamics of Y are given by

$$\begin{aligned}dY_t &= \kappa_Y^{\mathbb{P}} (\theta_Y^{\mathbb{P}} - Y_t) dt + \sigma_Y \sqrt{Y_t} dB_t^{\mathbb{P},(Y)} + dJ_t^{\mathbb{P},(Y)}, \\ dY_t &= \kappa_Y^{\mathbb{Q}} (\theta_Y^{\mathbb{Q}} - Y_t) dt + \sigma_Y \sqrt{Y_t} dB_t^{\mathbb{Q},(Y)} + dJ_t^{\mathbb{Q},(Y)}.\end{aligned}\tag{25}$$

Under \mathbb{P} , $B^{\mathbb{P},(Y)}$ is a Brownian motion and $J^{\mathbb{P},(Y)}$ an independent compound Poisson process. Under \mathbb{Q} , $B^{\mathbb{Q},(Y)}$ is a Brownian motion and $J^{\mathbb{Q},(Y)}$ an independent compound Poisson process.

To simplify notation, let

$$\begin{aligned}A_t^{\mathbb{P}} &= (a_{it}^{\mathbb{P}})_{1 \leq i \leq m} & A^{\mathbb{P}} &= (A_t^{\mathbb{P}})_{t \geq 0} \\ B_t^{\mathbb{P}} &= (b_{it}^{\mathbb{P}})_{1 \leq i \leq m} & B^{\mathbb{P}} &= (B_t^{\mathbb{P}})_{t \geq 0} \\ A_t^{\mathbb{Q}} &= (a_{it}^{\mathbb{Q}})_{1 \leq i \leq m} & A^{\mathbb{Q}} &= (A_t^{\mathbb{Q}})_{t \geq 0} \\ \mathbb{X}_t &= (X_{it})_{1 \leq i \leq m} & \mathbb{X} &= (\mathbb{X}_t)_{t \geq 0} \\ \Lambda_t^{\mathbb{P}} &= (\lambda_{it}^{\mathbb{P}})_{1 \leq i \leq m} & \Lambda^{\mathbb{P}} &= (\Lambda_t^{\mathbb{P}})_{t \geq 0} \\ \Lambda_t^{\mathbb{Q}} &= (\lambda_{it}^{\mathbb{Q}})_{1 \leq i \leq m} & \Lambda^{\mathbb{Q}} &= (\Lambda_t^{\mathbb{Q}})_{t \geq 0},\end{aligned}$$

and

$$\begin{aligned}\mathbb{Y} &= (Y_t)_{t \geq 0}, \\ \Theta_t^{\mathbb{P}} &= (\kappa_{Y,t}^{\mathbb{P}}, \theta_{Y,t}^{\mathbb{P}}, \sigma_{Y,t}, l_{Y,t}^{\mathbb{P}}, \mu_{Y,t}^{\mathbb{P}}, (\kappa_{i,t}^{\mathbb{P}}, \theta_{i,t}^{\mathbb{P}}, l_{i,t}^{\mathbb{P}}, \mu_{i,t}^{\mathbb{P}})_{1 \leq i \leq m}) \\ \Theta^{\mathbb{P}} &= (\Theta_t^{\mathbb{P}})_{t \geq 0} \\ \Theta_t^{\mathbb{Q}} &= (\kappa_{Y,t}^{\mathbb{Q}}, \theta_{Y,t}^{\mathbb{Q}}, \sigma_{Y,t}, l_{Y,t}^{\mathbb{Q}}, \mu_{Y,t}^{\mathbb{Q}}, (\kappa_{i,t}^{\mathbb{Q}}, \theta_{i,t}^{\mathbb{Q}}, l_{i,t}^{\mathbb{Q}}, \mu_{i,t}^{\mathbb{Q}})_{1 \leq i \leq m}) \\ \Theta^{\mathbb{Q}} &= (\Theta_t^{\mathbb{Q}})_{t \geq 0},\end{aligned}$$

¹⁸We briefly examined a model with time-fixed factor loadings in (23), but were unable to achieve an acceptable fit since the ratios $\lambda_{it}^{\mathbb{Q}}/\lambda_{it}^{\mathbb{P}}$ often vary dramatically over time. See also Section 7.1.

and

$$\begin{aligned}
\text{CDS}_t &= (c_{i,t,M}^{\text{cds}})_{1 \leq i \leq m, M \in \{5,7,10\}} & \text{CDS} &= (\text{CDS}_t)_{t \geq 0} \\
\text{TR}_t &= (c_{j,t,M}^{\text{tr}})_{1 \leq i \leq m, M \in \{5,7,10\}} & \text{TR} &= (\text{TR}_t)_{t \geq 0} \\
\text{IDX}_t &= (c_{t,M}^{\text{idx}})_{M \in \{5,7,10\}} & \text{IDX} &= (\text{IDX}_t)_{t \geq 0},
\end{aligned}$$

where $\Lambda^{\mathbb{P}}$ is the output of the model for physical default intensities estimated by Duffie, Saita, and Wang (2007) and Duffie, Eckner, Horel, and Saita (2009). Their model was fitted to a 25-year US corporate default history of 2,793 companies. (See Appendix A for details.) By taking the output of their model as given, we reduce the efficiency of our parameter estimation procedure compared to full maximum likelihood estimation, but avoid the computational complexity that a joint estimation would require.

We must conduct inference only about $\Theta \equiv (A^{\mathbb{P}}, B^{\mathbb{P}}, A^{\mathbb{Q}}, \Theta^{\mathbb{P}}, \Theta^{\mathbb{Q}})$ and \mathbb{Y} , since \mathbb{X} is then determined implicitly by $\Lambda^{\mathbb{P}}$, $A^{\mathbb{P}}$, $B^{\mathbb{P}}$ and \mathbb{Y} via (23). In the following, we therefore write $X_{it} = X_{it}(a_{it}^{\mathbb{P}}, b_{it}^{\mathbb{P}}, Y_t, \lambda_{it}^{\mathbb{P}})$ to emphasize that X_{it} is determined by the data and parameters in parentheses. For a fixed set of parameters Θ , using Bayes' rule and the Markov property for X_i and Y , we can write the likelihood function L_{Θ} of the data as

$$\begin{aligned}
&L_{\Theta}(\mathbb{Y}, \Lambda^{\mathbb{P}}, \text{CDS}, \text{TR}, \text{IDX}) && (26) \\
&\propto p_{\Theta}(\mathbb{Y}) p_{\Theta}(\Lambda^{\mathbb{P}}, \text{CDS}, \text{TR}, \text{IDX} \mid \mathbb{Y}) \\
&= p_{\Theta}(\mathbb{Y}) p_{\Theta}(\Lambda^{\mathbb{P}} \mid \mathbb{Y}) p_{\Theta}(\text{CDS}, \text{TR}, \text{IDX} \mid \mathbb{Y}) \\
&= p_{\Theta}(Y_0) \times \prod_{t=1}^T p_{bAJD}(Y_t \mid Y_{t-1}, \Theta_{t-1}^{\mathbb{P}}) \\
&\quad \times \prod_{t=1}^T \prod_{i=1}^m p_{bAJD}(X_{it}(a_{it}^{\mathbb{P}}, b_{it}^{\mathbb{P}}, Y_t, \lambda_{it}^{\mathbb{P}}) \mid X_{i,t-1}(a_{i,t-1}^{\mathbb{P}}, b_{i,t-1}^{\mathbb{P}}, Y_{t-1}, \lambda_{i,t-1}^{\mathbb{P}}), \Theta_{t-1}^{\mathbb{P}}) \\
&\quad \times \prod_{t=0}^T p_{\Theta}(\text{CDS}_t, \text{TR}_t, \text{IDX}_t \mid \mathbb{X}_t, Y_t),
\end{aligned}$$

where p_{bAJD} denotes the transition density of a basic AJD.

For the parameter estimation, we again impose the constraints (7)-(12) on the risk-neutral parameters, and in addition impose for each i and t ,

$$\begin{aligned}
\kappa_{i,t}^{\mathbb{P}} \theta_{it}^{\mathbb{P}} &= \kappa_{i,t}^{\mathbb{Q}} \theta_{it}^{\mathbb{Q}} \\
\kappa_{Y,t}^{\mathbb{P}} \theta_{Y,t}^{\mathbb{P}} &= \kappa_{Y,t}^{\mathbb{Q}} \theta_{Y,t}^{\mathbb{Q}} \\
a_{it}^{\mathbb{P}} &= \omega_{3,t} \lambda_{it}^{\mathbb{P}}.
\end{aligned} \tag{27}$$

The first two conditions in (27) correspond to a completely affine specification for the market-price of diffusive risk, and ensure that \mathbb{Q} is indeed an equivalent martingale measure for \mathbb{P} , see Appendix B for details. The last condition in (27) is used to arrive at

a parsimonious model. A singular value decomposition of the sample covariance matrix of $(\Delta\lambda_{it}^{\mathbb{P}} : 1 \leq i \leq m)$ gave a Spearman rank correlation of 0.91 between (i) the entries of the first eigenvector and (ii) average default intensities $\text{Avg}_t(\lambda_{it}^{\mathbb{P}})$, so that taking $a_i^{\mathbb{P}}$ proportional to $\lambda_i^{\mathbb{P}}$ is not overly restrictive.

Full maximum likelihood inference of the model parameters via

$$\hat{\Theta} = \arg \sup_{\Theta} L_{\Theta} (\mathbb{Y}, \Lambda^{\mathbb{P}}, \text{CDS}, \text{TR}, \text{IDX}),$$

for example using MCMC methods that treat \mathbb{X} and \mathbb{Y} as latent variables, would be desirable, but is computationally extremely burdensome. In addition, stale prices and market-microstructure noise in the CDS data would make the inference procedure prone to outliers. We therefore settled for a slightly less efficient estimation procedure of the model parameters, a mixture between maximum likelihood estimation and robust method of moments, which is computationally more tractable and gives robust parameter estimates. Before we turn to the actual fitting procedure, we need the following two results:

Proposition 1 *Let $\langle \lambda_i^{\mathbb{P}}, \lambda_j^{\mathbb{P}} \rangle (t)$ denote the continuous part of the quadratic covariation between $\lambda_i^{\mathbb{P}}$ and $\lambda_j^{\mathbb{P}}$ in the time interval $[0, t]$.¹⁹ Then*

$$(i) \quad \langle \lambda_i^{\mathbb{P}}, \lambda_j^{\mathbb{P}} \rangle (t) = \int_0^t a_{is}^{\mathbb{P}} a_{js}^{\mathbb{P}} \sigma_{Y,s}^2 Y_s ds.$$

(ii) *A consistent (for $\Delta t \rightarrow 0$) and model-free estimate of $\langle \lambda_i^{\mathbb{P}}, \lambda_j^{\mathbb{P}} \rangle (t)$ is given by*

$$\langle \lambda_i^{\mathbb{P}}, \lambda_j^{\mathbb{P}} \rangle (t) = \frac{1}{2} \{ \langle \lambda_i^{\mathbb{P}} + \lambda_j^{\mathbb{P}} \rangle (t) - \langle \lambda_i^{\mathbb{P}} \rangle (t) - \langle \lambda_j^{\mathbb{P}} \rangle (t) \},$$

where for a stochastic process X that is observed at times $\{0, \Delta t, 2\Delta t, \dots, n\Delta t = t\}$,

$$\langle X \rangle (t) = \sum_{i=1}^{n-1} |\Delta X_{i\Delta t}| |\Delta X_{(i+1)\Delta t}|$$

denotes its realized power variation.

The first property follows directly from the definition of physical default intensities via (23), (24) and (25), while the second property follows from the results in Barndorff-Nielsen and Shephard (2004).

¹⁹The derivative of this process can be thought of as the instantaneous Brownian covariance between $\lambda_i^{\mathbb{P}}$ and $\lambda_j^{\mathbb{P}}$.

Based on this corollary, we use

$$\widehat{Cov}_{cont}^{\mathbb{P}}(\lambda_{it}^{\mathbb{P}}, \lambda_{jt}^{\mathbb{P}}) \equiv \frac{\langle \lambda_i^{\mathbb{P}}, \lambda_j^{\mathbb{P}} \rangle(t) - \langle \lambda_i^{\mathbb{P}}, \lambda_j^{\mathbb{P}} \rangle(t-180)}{180}$$

as a rolling estimate of the continuous covariance between $\lambda_i^{\mathbb{P}}$ and $\lambda_j^{\mathbb{P}}$ at time t , where time is measured in days.

The fitting procedure is follows:

Algorithm 2

1. Fit \mathbb{X} , \mathbb{Y} , and $\Theta^{\mathbb{Q}}$ (including $\{\sigma_{Y,t} : 0 \leq t \leq T\}$) to the monthly times series of 5, 7 and 10-year CDS, credit index, and tranche spreads, again using the robust version (21) of tranche pricing errors.²⁰
2. The relation $a_{it}^{\mathbb{P}} = \omega_{3,t} \lambda_{it}^{\mathbb{P}}$ in (27) implies that

$$\begin{aligned} Cov_{cont}^{\mathbb{P}}(\lambda_{it}^{\mathbb{P}}, \lambda_{jt}^{\mathbb{P}}) &= a_{it}^{\mathbb{P}} a_{jt}^{\mathbb{P}} Cov_{cont}^{\mathbb{P}}(Y_t, Y_t) \\ &= \omega_{3,t}^2 \lambda_{it}^{\mathbb{P}} \lambda_{jt}^{\mathbb{P}} \sigma_{Y,t}^2 Y_t. \end{aligned}$$

Hence for $\mathcal{S} = \{(i, j) : 1 \leq i, j \leq m, i \neq j\}$,

$$median_{(i,j) \in \mathcal{S}} Cov_{cont}^{\mathbb{P}}(\lambda_{it}^{\mathbb{P}}, \lambda_{jt}^{\mathbb{P}}) = \omega_{3,t}^2 \sigma_{Y,t}^2 Y_t \cdot median_{(i,j) \in \mathcal{S}}(\lambda_{it}^{\mathbb{P}} \lambda_{jt}^{\mathbb{P}}),$$

so that a robust estimate of $\omega_{3,t}$ is

$$\widehat{\omega}_{3,t} = \sqrt{\frac{median_{(i,j) \in \mathcal{S}} \widehat{Cov}_{cont}^{\mathbb{P}}(\lambda_{it}^{\mathbb{P}}, \lambda_{jt}^{\mathbb{P}})}{\sigma_{Y,t}^2 Y_t \cdot median_{(i,j) \in \mathcal{S}}(\lambda_{it}^{\mathbb{P}} \lambda_{jt}^{\mathbb{P}})}}.$$

3. Calculate the factor loadings $a_{it}^{\mathbb{P}}$ and $b_{it}^{\mathbb{P}}$ of physical default intensities on the common and firm-specific factors, respectively, via

$$\begin{aligned} a_{it}^{\mathbb{P}} &= \omega_{3,t} \lambda_{it}^{\mathbb{P}} \\ b_{it}^{\mathbb{P}} &= \frac{\lambda_{it}^{\mathbb{P}} - a_{it}^{\mathbb{P}} Y_t}{X_{it}} = \frac{\lambda_{it}^{\mathbb{P}} (1 - \omega_{3,t} Y_t)}{X_{it}}. \end{aligned}$$

²⁰Excluding May 2005, the mean and standard deviation of the standardized residuals $\varepsilon_{Y,t+1} = \Delta Y_{t+1} / (\sigma_{Y,t} \sqrt{Y_t} \Delta t)$ are equal to -0.14 and 1.06, respectively. A Kolmogorov-Smirnov test against a standard normal distribution gave a p-value of 0.83. Hence, the volatility parameter time series $\sigma_{Y,t}$ is fairly consistent with the time-series behavior of Y , even though this property was not enforced in the estimation.

4. Given \mathbb{X} , \mathbb{Y} , $\{\sigma_{Y,t} : 0 \leq t \leq T\}$, and $\{\sigma_{X_i,t} : 1 \leq i \leq m, 0 \leq t \leq T\}$, fit $\Theta^{\mathbb{P}}$ via maximum likelihood estimation, taking into account the completely affine constraints (27). Due to the limited length of the dataset, we in addition impose the cross-sectional constraints, for all i ,

$$\begin{aligned}\kappa_i^{\mathbb{P}} &= \kappa_X^{\mathbb{P}} \\ l_i^{\mathbb{P}} &= l_X^{\mathbb{P}} \\ \mu_i^{\mathbb{P}} &= \mu_X^{\mathbb{P}},\end{aligned}$$

where $\kappa_X^{\mathbb{P}}$, $l_X^{\mathbb{P}}$ and $\mu_X^{\mathbb{P}}$ are constants. For the common factor Y , we set

$$\kappa_Y^{\mathbb{P}} = \frac{\ln(2)}{3} \approx 0.23,$$

so that the half-life of a systematic shock is equal to three years, which is at the upper end of half of the typical US business cycle length, see King and Watson (1996), and therefore leads to a conservative estimate of the market price of systematic risk in (28) below. Moreover, we set

$$\begin{aligned}l_Y^{\mathbb{P}} &= l_Y^{\mathbb{Q}} \\ \mu_Y^{\mathbb{P}} &= \mu_Y^{\mathbb{Q}},\end{aligned}$$

which leads to a conservative estimate of risk premia associated with the systematic factor.

Maximum likelihood estimation in Step 4 amounts to parameter inference about a discretely-observed basic AJD. See Appendix C for details.

Remark 2 Step 4 in Algorithm 2 relies on the assumption that the parameters governing physical default intensity dynamics do not change over time. In particular, companies have a stationary capital structure and the mechanism by which companies default does not exhibit any structural breaks. A change in legislation, like the Sarbanes-Oxley Act of 2002, might cause a permanent shift in the default pattern of corporations, everything else equal, so that this assumption may be violated.

6.2 Risk Premia

After specifying the joint model of physical and risk-neutral default intensity dynamics, we are able to describe the difference between the probability measures \mathbb{P} and \mathbb{Q} in terms of the Radon-Nikodym derivative $d\mathbb{Q}/d\mathbb{P}$. For the framework of Section 6.1, Appendix B provides an explicit formula for $d\mathbb{Q}/d\mathbb{P}$ and lists technical conditions under which \mathbb{Q} is indeed an *equivalent* martingale measure for \mathbb{P} . A convenient way to "summarize" the Radon-Nikodym derivative $d\mathbb{Q}/d\mathbb{P}$ is in terms of risk premia, namely:

- 1.) The market price of diffusive risk (systematic and firm-specific)
- 2.) The *pure* jump-to-default risk premium
- 3.) The correlation risk premium
- 4.) The market price of jump arrival risk
- 5.) The market price of jump size risk

Theorem 5 in Appendix B shows that the Radon-Nikodym derivative $d\mathbb{Q}/d\mathbb{P}$ can be written in terms of the above risk premia, whose definitions are made precise in the remainder of this section. Hence, there is no information "lost" by directly using these risk premia to evaluate investors' risk preferences in the credit markets.

6.2.1 Market Price of (Diffusive) Risk

The market price of risk measures the compensation that investors receive for being exposed to the risk that the realized value of a certain risk factor will turn out to be less favorable than expected. Let Z be a basic AJD with

$$\begin{aligned} dZ_t &= \kappa^{\mathbb{P}} (\theta^{\mathbb{P}} - Z_t) dt + \sigma \sqrt{Z_t} dB_t^{\mathbb{P}} + dJ_t^{\mathbb{P}} \\ dZ_t &= \kappa^{\mathbb{Q}} (\theta^{\mathbb{Q}} - Z_t) dt + \sigma \sqrt{Z_t} dB_t^{\mathbb{Q}} + dJ_t^{\mathbb{Q}}. \end{aligned}$$

Under \mathbb{P} , $B^{\mathbb{P},(Y)}$ is a Brownian motion and $J^{\mathbb{P},(Y)}$ an independent compound Poisson process with jump intensity $l^{\mathbb{P},(Z)}$ and exponentially distributed jumps with mean $\mu^{\mathbb{P},(Z)}$. Under \mathbb{Q} , $B^{\mathbb{Q},(Y)}$ is a Brownian motion and $J^{\mathbb{Q},(Y)}$ an independent compound Poisson process with jump intensity $l^{\mathbb{Q},(Z)}$ and exponentially distributed jumps with mean $\mu^{\mathbb{Q},(Z)}$. We call

$$\eta^{\text{MPR(Diff)}}(Z_t) = \frac{\kappa^{\mathbb{Q}}\theta^{\mathbb{Q}} - \kappa^{\mathbb{P}}\theta^{\mathbb{P}}}{\sigma\sqrt{Z_t}} + \frac{\kappa^{\mathbb{Q}} - \kappa^{\mathbb{P}}}{\sigma}\sqrt{Z_t}$$

the market price of diffusive risk for Z . The constraint $\kappa^{\mathbb{Q}}\theta^{\mathbb{Q}} = \kappa^{\mathbb{P}}\theta^{\mathbb{P}}$ of the completely affine risk premia specification implies that

$$\eta^{\text{MPR(Diff)}}(Z_t) = \frac{\kappa^{\mathbb{Q}} - \kappa^{\mathbb{P}}}{\sigma}\sqrt{Z_t}.$$

Hence, for the joint model of default intensities in Section 6.1, the market price of diffusive risk is

$$\eta^{\text{MPR(Diff)}}(X_{it}) = \frac{\kappa^{\mathbb{Q}} - \kappa^{\mathbb{P}}}{\sigma_i}\sqrt{X_{it}} = \frac{\kappa^{\mathbb{Q}} - \kappa^{\mathbb{P}}}{\sigma}\sqrt{\frac{X_{it}}{a_i^{\mathbb{Q}}}}$$

for the firm-specific risk factors and

$$\eta^{\text{MPR(Diff)}}(Y_t) = \frac{\kappa^{\mathbb{Q}} - \kappa^{\mathbb{P}}}{\sigma} \sqrt{Y_t}. \quad (28)$$

for the systematic risk factor.

6.2.2 Jump-to-Default and Correlation Risk Premium

In the framework of Section 6.1, risk-neutral default intensities can be rewritten as

$$\begin{aligned} \lambda_{it}^{\mathbb{Q}} &= X_{it} + a_{it}^{\mathbb{Q}} Y_t = \\ &= \frac{b_{it}^{\mathbb{P}}}{b_{it}^{\mathbb{P}}} X_{it} + \frac{a_{it}^{\mathbb{P}}}{b_{it}^{\mathbb{P}}} Y_t + \left(a_{it}^{\mathbb{Q}} - \frac{a_{it}^{\mathbb{P}}}{b_{it}^{\mathbb{P}}} \right) Y_t = \\ &= \frac{1}{b_{it}^{\mathbb{P}}} \lambda_{it}^{\mathbb{P}} + \left(a_{it}^{\mathbb{Q}} - \frac{a_{it}^{\mathbb{P}}}{b_{it}^{\mathbb{P}}} \right) Y_t. \end{aligned}$$

We call

$$\eta^{\text{pJTD}}(\lambda_{it}^{\mathbb{Q}}) = \frac{1}{b_{it}^{\mathbb{P}}} - 1$$

the *pure* jump-to-default (pJTD) risk premium and

$$\eta^{\text{Cor}}(\lambda_{it}^{\mathbb{Q}}) = \left(a_{it}^{\mathbb{Q}} - \frac{a_{it}^{\mathbb{P}}}{b_{it}^{\mathbb{P}}} \right) \quad (29)$$

the correlation risk premium of company i at time t . Under this convention, risk-neutral default intensities can be written as

$$\lambda_{it}^{\mathbb{Q}} = \lambda_{it}^{\mathbb{P}} + \eta^{\text{pJTD}}(\lambda_{it}^{\mathbb{Q}}) \lambda_{it}^{\mathbb{P}} + \eta^{\text{Cor}}(\lambda_{it}^{\mathbb{Q}}) Y_t, \quad (30)$$

reflecting instantaneous compensation for expected losses, *pure* jump-to-default risk and correlation risk.

Remark 3 Our definition of the JTD risk premium is slightly different than the one commonly used in the literature, which is $\lambda_{it}^{\mathbb{Q}}/\lambda_{it}^{\mathbb{P}} - 1$ (see for example Driessen (2005) Berndt, Douglas, Duffie, Ferguson, and Schranz (2005), Amato (2005) and Saita (2006)), since a multivariate model of credit risk allows one to further break down $\lambda_{it}^{\mathbb{Q}}/\lambda_{it}^{\mathbb{P}} - 1$ into compensation for pure JTD risk and correlation risk. See also Collin-Dufresne, Goldstein, and Helwege (2003) for a non-doubly-stochastic framework where $\lambda_{it}^{\mathbb{Q}}/\lambda_{it}^{\mathbb{P}} - 1$ can be decomposed into compensation for pure JTD risk (called timing risk in the paper) and contagion risk.

Remark 4 Definition (29) of the correlation risk premium is warranted, since in a doubly-stochastic setting, default time correlation is entirely attributable to correlation across firms’ levels of future credit-worthiness, which in our case is driven by the systematic factor Y . In the presence of contagion, influence would also be able to ”flow” in the opposite direction, in the sense that the default of a particular firm can impact the joint future credit-worthiness of surviving companies above and beyond economic conditions. In this case, the correlation risk premium would consist of compensation for (i) correlation of future credit-worthiness due to joint dependence on economic conditions and (ii) default contagion.

6.2.3 Market Price of Jump Arrival Risk

The market price of jump arrival risk is defined as

$$\eta^{\text{MPR(JA)}}(X_{it}) = \frac{l_i^{\text{Q}}}{l_{X_i}^{\text{P}}}$$

for the idiosyncratic risk factors. Recall that due to the limited length of our dataset, we assumed in Section 6.1 that $\eta^{\text{MPR(JA)}}(Y_t)$ is equal to zero, so that the market price of jump arrival risk for the common factor is subsumed into the correlation risk premium.

6.2.4 Market Price of Jump Size Risk

The market price of jump size risk is defined as

$$\eta^{\text{MPR(JS)}}(X_{it}) = \frac{\mu_i^{\text{Q}}}{\mu_{X_i}^{\text{P}}}$$

for the idiosyncratic risk factors. Recall that due to the limited length of our dataset, we assumed in Section 6.1 that $\eta^{\text{MPR(JS)}}(Y_t)$ is equal to zero, so that the market price of jump size risk for the common factor is subsumed into the correlation risk premium.

7 Results – Risk Premia

This sections presents the fit of the joint model for physical and risk-neutral default intensities. It also describes a decomposition of credit tranche spreads into compensation for expected losses and various types of risk premia.

7.1 Case Study – Alcoa Inc.

We start with a short case study to illustrate the differences between physical and risk-neutral default intensities. For this purpose, we picked Alcoa Inc. (ticker symbol

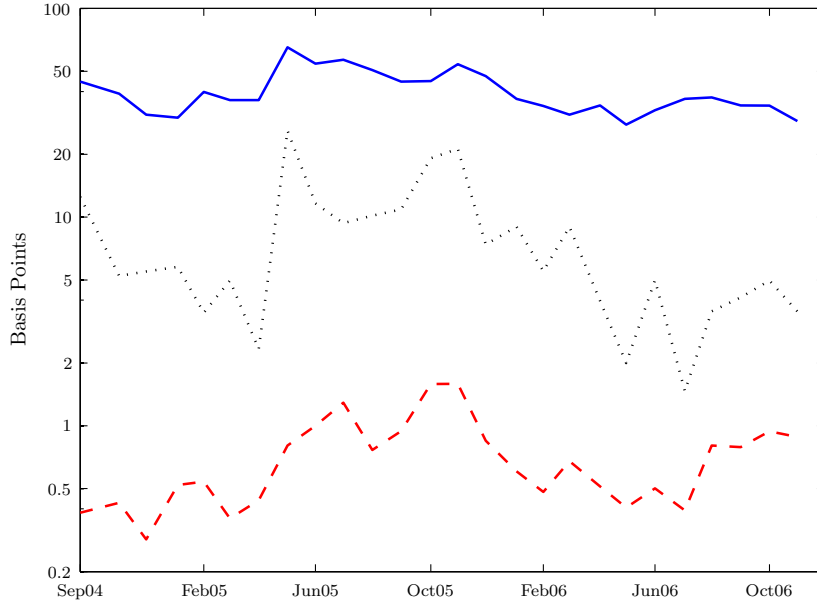


Figure 6: Evolution of default risk for Alcoa Inc. at a monthly frequency: 5-year CDS spread divided by 0.6 (solid line), risk-neutral default intensity λ_{it}^Q (dotted line), and physical default intensity λ_{it}^P (dashed line). Risk-neutral default intensities are from the fitted basic AJD model, while physical default intensities are from the Cox proportional hazards model estimated by Duffie, Saita and Wang (2007), and Duffie, Eckner, Horel and Saita (2006).

AA), which as of July 2007 had a market capitalization of \$36bn and had maintained a credit rating of 'A' between September 2004 and November 2006. The historical one-year default probability for this rating category is 0.08% according to Moody's Investor Service, "Historical Default Rates of Corporate Bond Issuers, 1920-1999".

Figure 6 shows the fitted physical and risk-neutral default intensity for Alcoa Inc. between September 2004 and November 2006, together with the 5-year CDS spread divided by 0.6, which for low-risk issuers is a rough estimate of the average risk-neutral yearly default rate over the next five years. As expected, physical and risk-neutral default intensities are highly correlated, having a sample correlation of 0.62, while the sample correlation of monthly changes in intensities is 0.48. We also see that the risk-neutral default intensity has been consistently higher than the physical default intensity, indicating that jump-to-default and/or correlation risk is priced in the decomposition (30).²¹ We also see that the ratio of the risk-neutral to physical default intensity λ_i^Q/λ_i^P

²¹For about 90% of the companies in our dataset, this pattern holds at each single point in time

can vary considerably over time. For example, between May 2005 and November 2006 this ratio dropped from a value around 20 to a value around 5, which indicates that in our framework it would be difficult to fit a model with constant factor loadings in (23) to the time series of physical and risk-neutral default intensities.²² Finally, we see that the average risk-neutral yearly default rate over the next five years is consistently higher than the risk-neutral default intensity $\lambda_i^{\mathbb{Q}}$, reflecting the typical upward-sloping term-structure of default rates for investment-grade rated issuers.

7.2 Fitted Parameters

The fitted risk-neutral model parameters $\Theta^{\mathbb{Q}}$ from Step 1 in Algorithm 2 are similar to those in Section 5, and therefore not reported here. Idiosyncratic credit risk shocks are highly mean reverting under the physical probability measure, with $\kappa_X^{\mathbb{P}} = 1.34$. Moreover, idiosyncratic jumps are more frequent ($l_X^{\mathbb{P}} = 0.992$) and smaller ($\mu_X^{\mathbb{P}} = 12.5$ basis points) under the physical probability measure than under the risk-neutral probability measure.

Figure 7 shows the time series of the median ratios $a_{it}^{\mathbb{P}}/a_{it}^{\mathbb{Q}}$ and $b_{it}^{\mathbb{P}}/b_{it}^{\mathbb{Q}}$, where the median is taken at each point in time over the set of CDX.NA.IG members on that day. We see that $b_{it}^{\mathbb{P}}/b_{it}^{\mathbb{Q}}$ is consistently less than one between September 2004 and November 2006, indicating that for the median firm, the *pure* jump-to-default risk premium is positive in the decomposition (30) of risk-neutral default intensities. Most of the time the median ratio $a_{it}^{\mathbb{P}}/a_{it}^{\mathbb{Q}}$ is smaller than the median ratio $b_{it}^{\mathbb{P}}/b_{it}^{\mathbb{Q}}$, so that the median correlation risk premium (29) is positive. Only around May 2005 did the correlation risk premium turn significantly negative.

7.3 Tranche Spread Decomposition

The joint model for physical and risk-neutral default intensities allows one to decompose credit index and tranche spreads into compensation for expected losses and the risk premia described in Section 6.2. To this end, we combine the market price of diffusive, jump-arrival, and jump size risk ($\eta^{\text{MPR(Diff)}}(X_{it})$, $\eta^{\text{MPR(JA)}}(X_{it})$ and $\eta^{\text{MPR(JS)}}(X_{it})$, respectively) for the idiosyncratic factors X_i and simply call

$$\eta^{\text{MPR}}(X_{it}) = \{\eta^{\text{MPR(Diff)}}(X_{it}), \eta^{\text{MPR(JA)}}(X_{it}), \eta^{\text{MPR(JS)}}(X_{it})\}$$

the market price of firm-specific risk. The market price of diffusive risk and the market price of jump risk are highly collinear, since risk-neutral drifts and jump parameters are highly collinear, and are therefore poorly identified separately.

between September 2004 and November 2006. For the remaining 10% of the companies, the pattern still holds most of the time.

²²On May 2, 2005, and November 1, 2006, the share price of Alcoa Inc. was 29.19 and 28.33, distance-to-default 5.93 and 5.59, and the five-year CDS spread 39 bps and 17 bps, respectively.

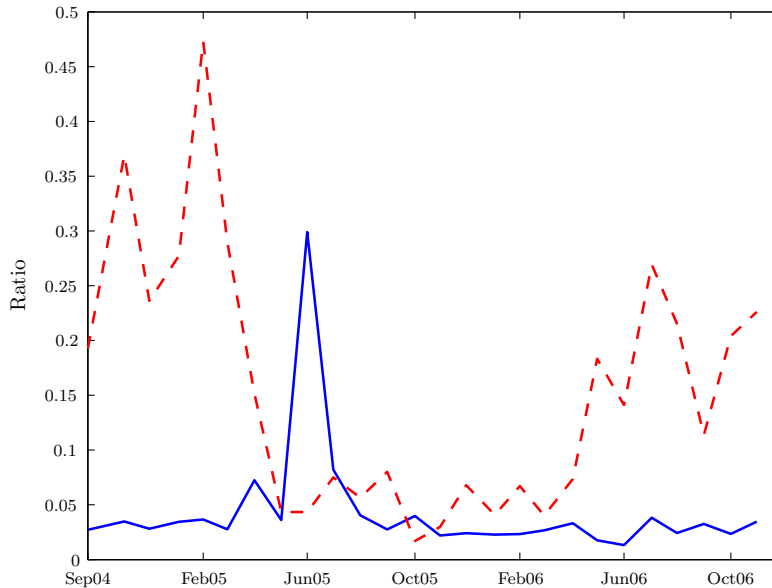


Figure 7: Median ratio a_{it}^P/a_{it}^Q (solid line) and b_{it}^P/b_{it}^Q (dashed line), where the median is taken at each point in time over the set of current CDX.NA.IG members.

Hence, we are left with four different risk premia and the following intuitive interpretations:

Market Price of Systematic Risk: Measures the amount of expected return that investors are willing to give up in order to guarantee that the economy develops as expected.

Market Price of Firm-Specific Risk: Given the market-wide performance, this measures the amount of expected return that investors are willing to give up in order to guarantee that a certain company performs in-line with the economy.

Correlation Risk Premium: Measures the amount of expected return that investors are willing to give up to guarantee that the future credit-worthiness of a certain company is uncorrelated (as opposed to positively correlated) with the future market-wide credit environment.

Pure Jump-to-Default Risk Premium: Measures investors' compensation for the uncertainty associated with the timing of default events, given both market-wide

and firm-specific risk factors that determine default probabilities.²³

The decomposition of tranche spreads into compensation for different sources of risk is now obtained as follows:

Expected losses: Set the risk premia $\eta^{\text{MPR}}(X_{it})$, $\eta^{\text{pJTD}}(X_{it})$, $\eta^{\text{Cor}}(X_{it})$ for $1 \leq i \leq m$, and $\eta^{\text{MPR(Diff)}}(Y_t)$ equal to zero. Recalculate credit index and tranche spreads. The ratio of the new spread to the original spread represents the fraction of the spread that is compensation for expected losses.

Risk Premia: The remaining fraction of credit spreads is distributed among (i) the market price of systematic risk, (ii) the market price of idiosyncratic risk (iii) the *pure* jump-to-default risk premium and (iv) the correlation risk premium. The proration is done by setting these risk premia equal to zero, one at a time, recalculating index and tranche spreads, and setting the spread contribution of each risk premium proportional to the resulting reduction in index and tranche spreads.²⁴

Figure 1 in the introduction shows the resulting decomposition of credit index and tranche spreads for the 5-year CDX index on November 1, 2006, which is the most recent available date in our dataset. As expected, the equity tranche up-front payment mainly represents compensation for expected losses due to defaults, compensation for the risk of firm-specific credit deterioration, and compensation for the uncertainty associated with the timing of defaults. As one moves up the capital structure, the fraction of tranche spreads that is compensation for the risk of a market-wide credit deterioration and compensation for correlation risk increases, and in case of the senior tranche accounts for more than 95% of the total spread. As a consequence, senior mezzanine and senior tranche spreads are likely to be much more sensitive to changes in investors' risk preferences than to changes in expected losses. This view has been expressed by many researchers and practitioners (see Tavakoli (2003)), but to the best of our knowledge has not previously been quantified.

In a manner analogous to the tranche spread decomposition above, one can obtain a decomposition of risk-neutral expected portfolio losses into compensation for different sources of risks, and attribute these components to individual tranches. For the time period from September 2004 to November 2006, Table 4 shows sample averages of the fractions of risk-neutral expected portfolio losses that correspond to individual

²³Compensation for correlation risk premium and the *pure* jump-to-default risk would not be separately identifiable in a univariate setting.

²⁴Alternatively, one can sequentially add the risk premia. We found the resulting decomposition of tranche spreads to be almost identical to the currently used decomposition. However, it is not clear in which order the individual risk premia should be added.

source-of-risk/tranche pairs. The last column and row show the total fraction of risk-neutral expected portfolio losses that can be attributed the individual sources of risks and tranches, respectively. Note that the total sum of the fractions is slightly less than one, since the super senior (30 - 100%) tranche is not included in Table 4 due to lack of available data. We see that by far the largest component of risk-neutral expected portfolio losses is compensation to the equity tranche for idiosyncratic risk (28%), followed by compensation to the equity tranche for pure JTD risk (22.7%), followed by compensation to the equity tranche for actual expected losses (19.7%), and then followed by compensation to the junior mezzanine tranche for systematic risk (4.2%). We also see that compensation to senior tranches is only a small fraction (<2%) of risk-neutral expected portfolio losses, and that this fraction is mostly compensation for systematic and correlation risk.

	0% - 3%	3% - 7%	7% - 10%	10% - 15%	15% - 30%	Sum
Expected Losses	0.197 (0.090)	0.010 (0.008)	0.002 (0.004)	0.001 (0.003)	0.000 (0.001)	0.211 (0.097)
Systematic Risk	0.033 (0.016)	0.042 (0.020)	0.010 (0.008)	0.007 (0.005)	0.010 (0.003)	0.102 (0.046)
Idiosyncratic Risk	0.280 (0.107)	0.037 (0.018)	0.002 (0.001)	0.001 (0.001)	0.001 (0.001)	0.321 (0.123)
Pure JTD Risk	0.227 (0.111)	0.034 (0.018)	0.002 (0.001)	0.001 (0.000)	0.001 (0.001)	0.265 (0.123)
Correlation Risk	0.033 (0.044)	0.034 (0.050)	0.010 (0.014)	0.007 (0.011)	0.008 (0.021)	0.092 (0.135)
Sum	0.770 (0.089)	0.157 (0.048)	0.027 (0.022)	0.018 (0.014)	0.020 (0.018)	0.992 (0.014)

Table 4: Attribution of risk-neutral expected portfolio losses to tranches and sources of risk. Entry (i, j) is the fraction of risk-neutral expected portfolio losses that is compensation of risk source i to tranche j . Data are sample averages for the 5-year CDX North-America Investment-Grade index between September 2004 and November 2006. Sample standard deviations are given parenthetically.

Remark 5 In an analogous manner, such a decomposition can be achieved for other credit derivatives, like CDS contracts or first-to-default baskets.

Remark 6 Since the number of defaults in our sample period (September 2004 to November 2006) was unusually low by historical standards, the contribution of expected losses to tranches spreads is likely underestimated in Figure 1 and Table 4. It would be desirable to repeat our analysis for a period that covers a full economic cycle, as soon as these data become available.

7.4 Tranche Vs. CDS Risk Characteristics

Coval, Jurek, and Stafford (2007) argue that certain structured finance instruments, such as a senior CDO tranche, offer far less compensation than alternative pay-off profiles, such as a short position in a 50% out-of-the-money put option on the S&P500 index. They argue that this mispricing is due to the tendency of rating agencies to pay attention only to expected losses when assigning ratings, but not to in what state of the economy these losses are going to occur. Under this presumption, issuing CDO tranches emerges as a mechanism for exploiting investors who solely rely on ratings for investment decisions.

To examine this point further, Table 5 provides the credit spread decomposition on November 1, 2006, for (i) the senior mezzanine tranche (10%-15%) of the 5-year CDX North-America Investment-Grade Index and (ii) a 5-year CDS contract for Wal-Mart Stores, Inc., which had a AA credit rating on this particular date. With a spread of 7.0 and 7.7 basis points, respectively, the implied risk-neutral probability of a loss to the seller of protection is extremely low for these two credit derivatives. The first row in Table 5 shows that the fraction of the credit spread that is compensation for expected losses, is about twice as high for the CDS contract compared to the CDX tranche contract.²⁵ In other words, for a given amount of expected losses, a seller of protection for a senior mezzanine CDX tranche can get an expected return that is roughly twice as high as those of a seller of protection for a CDS contract with similar pay-off profile. Hence, we do not find evidence that investors are fully oblivious about the differences in the risks between structured and single-name credit products. Further research in this area is required, to asses whether investors are at least partially oblivious about these differences, as argued by Coval, Jurek, and Stafford (2007).

	CDX: Senior Mezzanine Tranche	CDS: Wal-Mart Stores, Inc.
Expected Losses	0.005	0.010
Systematic Risk	0.355	0.154
Idiosyncratic Risk	0.042	0.137
Pure JTD Risk	0.040	0.451
Correlation Risk	0.558	0.248

Table 5: Fraction of spread that is compensation for different sources of risk. Data are for November 1, 2006, for (i) the senior mezzanine tranche (10% - 15%) of the 5-year CDX North-America Investment-Grade Index and (ii) the 5-year CDS spread of Wal-Mart Stores, Inc.

²⁵A similar relationship seems to hold for other highly rated issuers in the CDX portfolio. Of course, these results depend on a range of modeling assumption.

8 Conclusion

The emphasis of this paper has been two-fold. First, we conducted an empirical analysis of an extensive dataset of CDS, credit index, and credit tranche spreads between September 2004 and November 2006. To these data, we fitted an intensity-based model for the joint behavior of corporate default times, along the lines of Duffie and Gârleanu (2001), Mortensen (2006), and Eckner (2009). We found that the largest 5-year CDX.NA.IG tranche pricing errors occurred shortly after the introduction of the CDX index and tranche contracts, but that these discrepancies largely disappeared within a couple of months, either reflecting an evolution towards a more efficient market for structured credit derivatives, or more homogeneity in the models applied in industry practice. However, this downward trend in tranche pricing errors was interrupted by a sharp increase around May 2005, an episode in the credit market that is now commonly referred to as the "GM downgrade", or the "correlation crunch".

Second, we introduced a joint model for physical and risk-neutral default intensities, which we fitted to the time series of credit-default-swap, credit index, and credit tranche spreads between September 2004 and November 2006, as well as to the physical default intensity output from Duffie, Saita, and Wang (2007), and Duffie, Eckner, Horel, and Saita (2009). We presented a decomposition of risk-neutral expected portfolio losses into compensation for different sources of risks, and attributed these components to individual tranches. For our sample period we found that investors in the CDX first-loss tranche on average bear about 93.3% of the expected losses on the underlying portfolio of corporate debt, and receive about 77.0% of the total compensation for bearing portfolio losses. On the other hand, investors in the senior tranche, on average, bear about 0.2% of expected default losses on the underlying portfolio, and receive about 2% of the total compensation. In terms of the ratio, on average, of compensation rate per unit of expected loss, the first-loss piece carries a risk premium of 3.9, whereas the most senior tranches offer a risk premium of about 50. This extreme difference in risk premia is natural given that the most senior tranches bear losses only when there is a significant "meltdown" in corporate performance.

Although Coval, Jurek, and Stafford (2007) argued that investors in senior CDO tranches underprice systematic risks by solely relying on credit ratings for the purpose of making investment decisions, we found that investors are reasonably aware about the differences in the risks between structured and single-name credit products. However, in our framework we cannot preclude that investors are at least partially oblivious about these differences, and further research in this area would be desirable.

Finally, we broke down the compensation for bearing default risk into several components, namely compensation for actual expected losses, systematic risk, firm-specific risk, correlation risk, and *pure* jump-to-default (JTD) risk. We showed that various tranches have different rates of compensation per unit of expected loss because they carry these various types of risks in different proportions to each other. For our sample

period, the equity tranche is allocated 81.7% of the total compensation for firm-specific risk in the underlying portfolio of debt, but carries only 32.0% of the total compensation for systematic risk. The senior tranche, on the other hand, is allocated only 0.2% of the total compensation for firm-specific risk, but carries a relatively large fraction, 9.9%, of the total compensation for systematic risk. We hope that this breakdown will prove valuable to investors in single-name and structured credit products, who would like to better understand the types of risks to which they are exposed to, and to quantify the extent to which they are being compensated for these risks.

Appendices

A Cox Proportional Hazards Model for $\lambda^{\mathbb{P}}$

Using 25 years of US corporate default history on 2,793 companies, Duffie, Saita, and Wang (2007) and Duffie, Eckner, Horel, and Saita (2009) estimated a Cox proportional hazards model for physical default intensities. The default intensity of firm i at time t is given by

$$\lambda_{it}^{\mathbb{P}} = e^{\beta \cdot W_{it}},$$

where β is a parameter vector, and W_{it} a vector of firm-specific and macroeconomic covariates. The covariates are (i) distance-to-default, which is a volatility-adjusted measure of leverage, (ii) the firm's trailing 1-year stock return, (iii) the 3-month Treasury bill rate, and (iv) the trailing 1-year return on the S&P 500 index. See Duffie, Saita, and Wang (2007) and Duffie, Eckner, Horel, and Saita (2009) for a detailed description of the covariates and the maximum likelihood estimate of the parameter vector β . Not surprisingly, distance-to-default is the most influential covariate for determining default intensities, but the other covariates are statistically and economically significant as well.

B Equivalent Martingale Measure Conditions

This section provides technical conditions for the dynamics of $\lambda^{\mathbb{P}}$ under \mathbb{P} , and $\lambda^{\mathbb{Q}}$ under \mathbb{Q} , that ensure that \mathbb{Q} is indeed an equivalent martingale measure for \mathbb{P} . Standard references on equivalent martingale measures and risk-neutral pricing include Harrison and Kreps (1979) for a discrete time setting, Harrison and Pliska (1981) for the Brownian motion setting, Brémaud (1981) for point processes, Dai and Singleton (2000) for the completely affine market price of risk specification, and Cheridito, Filipovic, and Kimmel (2005) and Cheridito, Filipovic, and Yor (2005) for more general affine processes and changes of measure.

In the framework of Section 6, physical and risk-neutral default intensities are given by

$$\begin{aligned}\lambda_{it}^{\mathbb{Q}} &= X_{it} + a_i Y_t \\ \lambda_{it}^{\mathbb{P}} &= \lambda_{it}^{\mathbb{Q}} / (1 + \eta_{it}^{\text{JTD}}),\end{aligned}$$

for $1 \leq i \leq m$, where the jump-to-default risk premium η_i^{JTD} can represent compensation for *pure* JTD risk and for correlation risk, see decomposition (30). To simplify notation, let $Z = (Z_1, \dots, Z_{m+1}) = (X_1, \dots, X_m, Y)$ denote the vector of stochastic processes

driving default intensities. The dynamics of Z_j are given by

$$\begin{aligned} dZ_{jt} &= (\kappa_0^{\mathbb{P},(Z_j)} + \kappa_1^{\mathbb{P},(Z_j)} Z_{jt}) dt + \sigma_{Z_j} \sqrt{Z_{jt}} dB_t^{\mathbb{P},(Z_j)} + dJ_t^{\mathbb{P},(Z_j)} \\ dZ_{jt} &= (\kappa_0^{\mathbb{Q},(Z_j)} + \kappa_1^{\mathbb{Q},(Z_j)} Z_{jt}) dt + \sigma_{Z_j} \sqrt{Z_{jt}} dB_t^{\mathbb{Q},(Z_j)} + dJ_t^{\mathbb{Q},(Z_j)}. \end{aligned} \quad (31)$$

Under \mathbb{P} , $B^{\mathbb{P},(Z_j)}$ are independent Brownian motions, and $J^{\mathbb{P},(Z_j)}$ are independent compound Poisson processes with jump intensities $l^{\mathbb{P},(Z_j)}$ and exponentially distributed jumps with mean $\mu^{\mathbb{P},(Z_j)}$. Under \mathbb{Q} , $B^{\mathbb{Q},(Z_j)}$ are independent Brownian motions, and $J^{\mathbb{Q},(Z_j)}$ are independent compound Poisson processes with jump intensities $l^{\mathbb{Q},(Z_j)}$ and exponentially distributed jumps with mean $\mu^{\mathbb{Q},(Z_j)}$. The completely affine risk premia specification imposes the parameter constraints, for each j ,

$$\kappa_0^{\mathbb{P},(Z_j)} = \kappa_0^{\mathbb{Q},(Z_j)}.$$

We next show that the density (Radon-Nikodym derivative)

$$\Lambda_t = E \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathcal{F}_t \right)$$

can be written as a product of factors representing (i) changes in the drift of the $m+1$ Brownian motions, (ii) changes in the $m+1$ jump intensities and jump size distributions, and (iii) scaling of the m intensities $\lambda_i^{\mathbb{P}}$ by one plus the jump-to-default risk premium. Specifically,

$$\begin{aligned} \Lambda_t &= \left(\prod_{j=1}^{m+1} \Lambda_{jt}^{(1)} \right) \left(\prod_{j=1}^{m+1} \Lambda_{jt}^{(2)} \right) \left(\prod_{i=1}^m \Lambda_{it}^{(3)} \right) \\ &= \left(\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} \right)_t \left(\frac{d\hat{\mathbb{Q}}}{d\tilde{\mathbb{Q}}} \right)_t \left(\frac{d\mathbb{Q}}{d\hat{\mathbb{Q}}} \right)_t. \end{aligned} \quad (32)$$

The first two factors in (32) represent the change in dynamics of Z_j under \mathbb{P} to Z_j under \mathbb{Q} , while the last term reflects the change from $\lambda_i^{\mathbb{P}}$ under \mathbb{Q} to $\lambda_i^{\mathbb{Q}}$ under \mathbb{Q} . The remainder of this Appendix provides technical conditions for (32) to define an arbitrage-free pricing measure \mathbb{Q} equivalent to \mathbb{P} .

The following proposition governs the drift change of the stochastic processes Z_j . See Dai and Singleton (2000) for details.

Proposition 2 *Fix a time horizon $T > 0$. Assume that, for each j ,*

$$\kappa_0^{\mathbb{P},(Z_j)} = \kappa_0^{\mathbb{Q},(Z_j)}.$$

Define the market price of (diffusive) risk for Z_j as

$$\eta^{\text{MPR(Diff)}}(Z_{jt}) = \frac{\kappa_1^{\mathbb{Q},(Z_j)} - \kappa_1^{\mathbb{P},(Z_j)}}{\sigma_{Z_j}} \sqrt{Z_{jt}}.$$

Then $\tilde{\mathbb{Q}}_T$, defined by

$$E \left(\frac{d\tilde{\mathbb{Q}}_T}{d\mathbb{P}} \mid \mathcal{F}_t \right) = \prod_{j=1}^{m+1} \exp \left(- \int_0^t \eta^{\text{MPR(Diff)}}(Z_{js}) dB_s^{\mathbb{P},(Z_j)} - \frac{1}{2} \int_0^t [\eta^{\text{MPR(Diff)}}(Z_{js})]^2 ds \right),$$

is a probability measure equivalent to \mathbb{P} . Moreover, under $\tilde{\mathbb{Q}}_T$ the dynamics of Z_j for $1 \leq j \leq m+1$ are the same as under \mathbb{P} except that the mean-reversion coefficient $\kappa_1^{\mathbb{P},(Z_j)}$ in (31) is replaced by $\kappa_1^{\mathbb{Q},(Z_j)}$.

The following proposition governs the change in the jump intensity and jump size distribution of the stochastic processes Z_1, \dots, Z_{m+1} . See Cheridito, Filipovic, and Yor (2005) for a similar formulation.

Proposition 3 Fix a time horizon $T > 0$. Let $\nu_j^{\mathbb{P}}(z, \cdot)$ and $\nu_j^{\mathbb{Q}}(z, \cdot)$ denote the jump size distribution of Z_j under \mathbb{P} and \mathbb{Q} , respectively. Assume that the jump intensities under \mathbb{P} and \mathbb{Q} are bounded away from zero and bounded above, that is

$$\begin{aligned} K_1 &< l^{\mathbb{P},(Z_j)} < K_1 \\ K_2 &< l^{\mathbb{Q},(Z_j)} < K_2 \end{aligned}$$

for some positive constants K_1 and K_2 . Define

$$\phi_j(z, \zeta) = \frac{\nu_j^{\mathbb{Q}}(z, \zeta) l^{\mathbb{Q},(Z_j)}}{\nu_j^{\mathbb{P}}(z, \zeta) l^{\mathbb{P},(Z_j)}}$$

and

$$\gamma_j(z, \zeta) = \phi_j(z, \zeta) \log \phi_j(z, \zeta) - \phi_j(z, \zeta) + 1.$$

Assume that, for some finite constant K_3 ,

$$\int_0^\infty \gamma_j(z, \zeta) l^{\mathbb{P},(Z_j)} \nu_j^{\mathbb{P}}(z, d\zeta) < K_3 \tag{33}$$

for $1 \leq j \leq m+1$. Then $\hat{\mathbb{Q}}_T$, defined by

$$E \left(\frac{d\hat{\mathbb{Q}}_T}{d\tilde{\mathbb{Q}}_T} \mid \mathcal{F}_t \right) = \prod_{j=1}^{m+1} \exp \left(\int_0^t \int_{\mathbb{R}} \gamma_j(z, \zeta) l^{\mathbb{P},(Z_j)} \nu_j^{\mathbb{P}}(Z_{js}, d\zeta) ds \right)$$

is a probability measure equivalent to $\tilde{\mathbb{Q}}_T$. Moreover, under $\hat{\mathbb{Q}}_T$ the dynamics of Z_j for $1 \leq j \leq m+1$ are the same as under $\tilde{\mathbb{Q}}_T$, except that the jump intensity and jump size distribution are now given by $l^{\mathbb{Q},(Z_j)}$ and $\nu_j^{\mathbb{Q}}(z, \cdot)$, respectively.

Remark 7 *In the case of exponentially distributed jumps and constant jump intensities*

$$\phi_j(z, \zeta) = \mathbf{1}_{\{\zeta \geq z\}} \frac{l^{\mathbb{Q},(Z_j)} \mu^{\mathbb{P},(Z_j)}}{l^{\mathbb{P},(Z_j)} \mu^{\mathbb{Q},(Z_j)}} \exp \left(\left(\frac{1}{\mu^{\mathbb{P},(Z_j)}} - \frac{1}{\mu^{\mathbb{Q},(Z_j)}} \right) (\zeta - z) \right).$$

Hence (33) is satisfied if $l^{\mathbb{P},(Z_j)}$, $l^{\mathbb{Q},(Z_j)}$, $\mu^{\mathbb{P},(Z_j)}$ and $\mu^{\mathbb{Q},(Z_j)}$ are positive.

Propositions 2 and 3 govern the technical conditions for the change in dynamics of Z_j (and hence also $\lambda_i^{\mathbb{P}}$) under \mathbb{P} to the dynamics under \mathbb{Q} . The last step is a scaling of $\lambda_i^{\mathbb{P}}$ by one plus the jump-to-default risk premium η_i^{JTD} . Note that the \mathbb{Q} -dynamics of the physical default intensity $\lambda_i^{\mathbb{P}}$ and of the jump-to-default risk premium η_i^{JTD} are not directly observable from market data (as opposed to historical default data or CDS and tranche spreads). Only the \mathbb{Q} -dynamics of the product

$$\lambda_i^{\mathbb{P}} (1 + \eta_i^{\text{JTD}}) = \lambda_i^{\mathbb{Q}}$$

are observable, namely via the prices of various credit derivative contracts. Since the \mathbb{Q} -dynamics of $\lambda_i^{\mathbb{P}}$ and η_i^{JTD} are not separately identifiable in our framework, and we make the following assumption.

Assumption 2 *The \mathbb{Q} -dynamics of η_i^{JTD} for $1 \leq i \leq m$ are the same as the \mathbb{P} -dynamics. That is, there is no risk premium associated with the evolution of jump-to-default risk premium over time.*

We can now turn to the last step in the change of measure from \mathbb{P} to \mathbb{Q} . For details see Brémaud (1981) or Duffie (2001), Proposition F.5.

Proposition 4 *Fix a time horizon $T > 0$. Assume that $(1 + \eta_i^{\text{JTD}})$ is a strictly positive and predictable process with*

$$\int_0^T (1 + \eta_{it}^{\text{JTD}}) \lambda_{it}^{\mathbb{P}} dt < \infty \quad \hat{\mathbb{Q}}_T - \text{almost surely.}$$

Further assume that the local martingales

$$\Lambda_{it}^{(3)} = \exp \left(- \int_0^t \eta_{is}^{\text{JTD}} \lambda_{is}^{\mathbb{P}} ds \right) (\mathbf{1}_{\{\tau_i \leq t\}} (1 + \eta_{i,\tau_i}^{\text{JTD}}) + \mathbf{1}_{\{\tau_i > t\}})$$

are in fact $\hat{\mathbb{Q}}_T$ -martingales for $1 \leq i \leq m$ and $0 \leq t \leq T$. Then the probability measure \mathbb{Q}_T , defined by

$$E \left(\frac{d\mathbb{Q}_T}{d\hat{\mathbb{Q}}_T} \mid \mathcal{F}_t \right) = \prod_{i=1}^m \Lambda_{it}^{(3)}$$

is equivalent to $\hat{\mathbb{Q}}_T$. Moreover, the dynamics of $\lambda_i^{\mathbb{Q}} \equiv \lambda_i^{\mathbb{P}} \eta_i^{\text{JTD}}$ under \mathbb{Q}_T are the same as that of $X_i + a_i Y$ under \mathbb{Q}_T .

The following theorem combines the above propositions and assumptions.

Theorem 5 *Fix a time horizon $T > 0$. Suppose the conditions in Proposition 2, 3 and 4 are satisfied and that Assumption 2 holds. Then $(\Lambda_t : 0 \leq t \leq T)$, with Λ_t given by (32), is a strictly positive \mathcal{F}_t -martingale under \mathbb{P} , and Λ_T is the Radon-Nikodym derivative of a probability measure \mathbb{Q}_T equivalent to \mathbb{P} .*

For $0 \leq t \leq s \leq T$, the unique arbitrage-free price at time t for a security that pays one monetary unit at time s if $\tau_i > s$ (and zero otherwise) is

$$V_t = E^{\mathbb{Q}_T} \left(e^{-\int_t^s r_u du} \mathbf{1}_{\{\tau_i > s\}} \mid \mathcal{F}_t \right). \quad (34)$$

Equivalently,

$$\tilde{V}_t = P_t e^{\int_0^t r_u du}$$

is a martingale on $(\Omega, \mathcal{F}_T, (\mathcal{F}_t : 0 \leq t \leq T), \mathbb{Q}_T)$.

Proof Proposition 2 implies that $\tilde{\mathbb{Q}}_T$ is equivalent to \mathbb{P} . Proposition 3 implies that $\hat{\mathbb{Q}}_T$ is equivalent to $\tilde{\mathbb{Q}}_T$. Proposition 4 implies that \mathbb{Q}_T is equivalent to $\hat{\mathbb{Q}}_T$. We therefore also have that \mathbb{Q}_T is equivalent to \mathbb{P} and that $\Lambda_T > 0$ almost surely under \mathbb{P} . Since

$$\Lambda_t = E \left(\frac{d\mathbb{Q}_T}{d\mathbb{P}} \mid \mathcal{F}_t \right) = E (\Lambda_T \mid \mathcal{F}_t),$$

it follows that $\Lambda_t > 0$ almost surely under \mathbb{P} . The tower property implies that $(\Lambda_t : 0 \leq t \leq T)$ is an \mathcal{F}_t -martingale under \mathbb{P} . Finally, the security pricing formula (34) follows from the general theory of arbitrage-free pricing, see for example Section 6 in Duffie (2001).

□

Note that it is easy to generalize (34) to the pricing of securities with more complicated pay-off structures, see for example Section 6L in Duffie (2001).

C MLE of Physical Default Intensity Dynamics

This section provides details about Step 4 of Algorithm 2, which involves the estimation of the parameters governing physical default intensity dynamics.

Recall that $\{X_{it} : 1 \leq i \leq m, 0 \leq t \leq T\}$ and $\{Y_t : 0 \leq t \leq T\}$ are outputs of the fitting procedure in Section 4.2. Inference about the parameters governing the \mathbb{P} -dynamics of X_i and Y therefore amounts to estimating the parameters of a discretely observed basic AJD. For such a process Z with parameters $\eta = (\kappa, \theta, \sigma, l, \mu)$, let $p_{bAJD}(\cdot \mid z, \eta, s, t)$

denote its transition density between times s and time t for $Z_s = z$. Given a set of observations $\{Z_{t_1}, \dots, Z_{t_M}\}$, the likelihood function L_η is

$$L_\eta(Z_{t_1}, \dots, Z_{t_k}) = p(Z_{t_1} | \eta) \prod_{k=2}^M p_{bAJD}(Z_{t_k} | Z_{t_{k-1}}, \eta, t_{k-1}, t_k), \quad (35)$$

where $p(Z_{t_1} | \eta)$ the initial distribution of the process. We base inference about the basic AJD parameters on the time-discretization

$$\Delta Z_{j,t_k} = Z_{j,t_k} - Z_{j,t_{k-1}} = \kappa(\theta - Z_{t_{k-1}}) \Delta t_k + \sigma \sqrt{Z_{t_{k-1}} \Delta t_k} \varepsilon_{t_k} + \Delta J_{t_k}, \quad (36)$$

where $\Delta t_k = t_k - t_{k-1}$, ε_{t_k} are independent standard normal random variables, and ΔJ_{t_k} are independent random variables with distribution

$$\mathbb{P}(\Delta J_{t_k} \in [z, z + d\xi]) = \begin{cases} \exp(-l\Delta t_k), & \text{for } z = 0 \\ (1 - \exp(-l\Delta t_k)) \mu \exp(-\mu z) d\xi, & \text{for } z > 0 \end{cases}$$

The discretization (36) assumes that at most one jump can occur in each time interval $(t_{k-1}, t_k]$. In simulation studies, Eraker (2001) and Eraker, Johannes, and Polson (2003) show that when using daily or monthly data and parameters values similar to ours, the discretization bias due to using (36) is negligible.

Since X_1, \dots, X_m, Y are independent, parameter inference via (35) can be conducted separately for each process, for example, via Newton-Raphson maximization of the likelihood function. Nevertheless, it is easy to impose joint constraints on the parameters if desired.

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